NOTE TO THE EXAMINERS

In the preface, I thanked my supervisor, Dr M.F. Newman, for suggestions which led to an improved presentation of these results. The most far-reaching of these suggestions, made less than forty-eight hours before the first chapters were to be handed to the typist, has resulted in the complete removal of the previous Chapter 3, and the reduction of the previous Chapter 4, which made up half the bulk of the thesis, to the present Section 3.2 which is much easier to read, conceptually simpler, and a twentieth as long.

One consequence of the reorganisation is that Chapter 1 is left with a bulk out of proportion to its importance in the development of the argument. Another is that the present Chapters 3 and 4 have been organised and written in their present form within a space of one-and-a-half weeks. I apologise for any incoherency which remains as a result of this haste.

POWER AND COMMUTATOR STRUCTURE OF GROUPS

by

David Shield

A thesis submitted to the

Australian National University

for the degree of

Doctor of Philosophy

Canberra January, 1976

STATEMENT

The work reported in this thesis is my own, except where otherwise stated.

D. Shield.

PREFACE

The bounds on nilpotency class of group extensions reported in this thesis grow out of and complete the work reported in my MA thesis on Engel length of group extensions. A few of the lemmas presented here are closely related to some in that thesis, but the substance of the work reported here, and all the main results, are new.

Since the beginning of 1971 when I started this work, I have been Senior Tutor in the Department of Pure Mathematics in the Faculty of Arts of the Australian National University; and am grateful to Professor Hanna Neumann, to the several other members of the department who led it in the period following her death, and to its new head, Professor Neil Trudinger, for their friendship and guidance. I acknowledge also the support received in the form of an ANU Postgraduate Scholarship during 1975 while I have been on leave of absence from the Department of Pure Mathematics. Completion of this work has been helped greatly by the respite from teaching duties.

In particular I thank my supervisor, Dr M.F. Newman, for his guidance and encouragement, and for suggestions which have greatly improved the presentation of these results.

My deep gratitude is due also to my wife, Jenny, for her valiant efforts in bringing up our family almost single-handed over the past few months, and her willing additional help with proof-reading.

Finally, not only thanks but also congratulations to Mrs Barbara Geary, for her extraordinary ability to produce a typescript more accurate than the manuscript from which she worked.

ABSTRACT

Let G be a group with a normal subgroup H whose index is a power of a prime p, and which is nilpotent and has exponent a power of p. Gilbert Baumslag has shown that such a group is nilpotent; and that, conversely, if a non-trivial wreath product is nilpotent, then it satisfies the description above with H as its base group.

The main result of this thesis, in Theorem 3.3.3, is an upper bound on the nilpotency class of G in terms of parameters of H and G/H. Theorem 3.4.4 shows that this bound is attained whenever G is a wreath product and H its base group; thus the question of the nilpotency class of nilpotent wreath products is answered.

The answer involves a descending central series of subgroups, here called the cpp-series, previously defined independently by Jennings and Zassenhaus. The cpp-series seems more natural in the present context than does the lower central series; and the cpp-class of a nilpotent wreath product is also found. Definitions of these and other series grow out of the investigation in Chapter 1 of relationships between operations of commutation, raising to the pth power, and multiplication in groups.

Chapter 2 contains a construction for a basis for a finite p-group, such that the standard form of an element exhibits its position in the cpp-series.

The main results of Chapter 3 are applied in Chapter 4 to show that if certain restricted Burnside groups (whose existence is an open question) do in fact exist, then they are very large, and have such a high nilpotency class as to make it improbable that they can be computed.

CONTENTS

PREFACE		(ii)
ABSTRACT		(iii)
CONTENTS		(iv)
INTRODUCTION		1
	Background	1
	Outline	1
	Notation and terminology	3
CHAPTER 1	Words	4
CHAPTER 2	A Basis	75
CHAPTER 3	Group Extensions	83
CHAPTER 4	Lower Bounds on Some Potential "Restricted Burnside" Groups	104
REFERENCES		115

INTRODUCTION

0.1 Background

Gilbert Baumslag showed in §3 of [2] that a wreath product $A \text{ wr } B$ of two groups is nilpotent if and only if for some prime p the "bottom group" A is nilpotent with exponent a power of p and the "top group" B is a finite p-group. The proof of the "if" part, in Lemma 3.8 of [2], applies to group extensions in general, not only to wreath products. Although an upper bound on the class of such extensions is implicit in the proof of Lemma 3.8, it is not stated, and is extremely high.

Liebeck [16] found the exact nilpotency class of a nilpotent wreath product $A \text{ wr } B$ in the special case where A and B are both abelian, and observed that his result provides a lower bound to the class in the general case. Since then, exact results in other special cases and improved upper and lower bounds in general have been given by Teresa Scruton [26], J.D.P. Meldrum [18], and Larry Morley [19] and [20]. Robert Sandling [23] has pointed out that earlier work by S.A. Jennings [13] gives the exact class when A is cyclic of order p and B is an arbitrary finite p-group. Some of these results are discussed in more detail in subsection 3.4.6 of this thesis.

0.2 Outline

The main result of this thesis is an upper bound, in Theorem 3.3.3, on the class of those group extensions shown to be nilpotent by Gilbert Baumslag in Lemma 3.8 of [2]. That this bound is best possible is shown by the proof in Theorem 3.4.4 that the same number is a lower bound on the class of a wreath product satisfying the same conditions.

The tools which make these results possible are the idea of $w^{e}_{a,b}$-weight of a group element, and related results developed in Chapter 1, and the type of standard basis for a finite p-group constructed in Chapter 2.

The form of the calculations, and of the results obtained, makes it appear that in this context the lower central series of a group and the associated weight of a group element are less significant and natural than what are here called the cpp-series and the cpp-weight. If the cpp-series of a group reaches the trivial subgroup in finitely many steps, then the group is said to be cpp-nilpotent; and the integer used to label the last non-trivial subgroup in the series is called the cpp-class of the group. A group is cpp-nilpotent for a particular prime, p, if and only if it is nilpotent and has exponent a power of p. Definitions and more details are contained in Section 1.7.

Thus Gilbert Baumslag's Lemma 3.8, already mentioned above, may be stated: "For a given prime p, an extension of a cpp-nilpotent group by a finite cpp-nilpotent group is cpp-nilpotent". The cpp-class of such extensions is bounded above in Theorem 3.3.3, and the cpp-class of a wreath product is bounded below in Theorem 3.4.4, together with the corresponding results on nilpotency class.

Chapter 1 of this thesis deals with the operations of commutation, raising to the pth power and multiplication in groups by studying them in universal algebras constructed specifically for this purpose. Most of Sections 1.1 to 1.6 simply provide tools for use in the later chapters, but the results of Lemma 1.5.5 and Corollary 1.6.3 may have some independent interest, the former illustrated by an example in subsection 1.5.8.

In Chapter 2 a standard basis for a finite p-group is constructed. This basis has many desirable properties which prove very useful in later chapters. It might well have other applications in investigations involving finite p-groups, and possibly also their group rings.

The main results of Chapter 3 have already been mentioned. In Chapter 4, they are applied to show that if certain "restricted Burnside" groups whose existence has not been proved do in fact exist, then they are very large, and have such high nilpotency class as to make it unlikely that they can be computed by present techniques.

0.3 Notation and terminology

Elements of groups and algebras will be denoted by lower case Greek letters, such as α, β, φ, and ψ; mappings (and elements of operator sets in universal algebras) by smaller underlined Greek letters such as $\underline{\alpha}$, $\underline{\gamma}$ and $\underline{\pi}$; and integers and most integer-valued functions by lower case Roman letters such as a, b and f, though the integer $R(q, x, k)$ also occurs in 3.4.1.

The symbols Z, N, and Z^+ respectively represent the sets of all integers, of all non-negative integers, and of all positive integers. For l in Z^+, the underlined symbol $\underline{l}$ denotes the set

$$\underline{l} = \{i \in \mathrm{N} : 0 \leq i < l\} .$$

Upper case German script, here represented by double underlining as in $\underline{\underline{V}}$, is used to represent varieties. In particular, for n in Z^+, the symbols $\underline{\underline{A}}_n$ and $\underline{\underline{B}}_n$ represent respectively the variety of all abelian groups of exponent n and of all groups of exponent n.

Other symbols are defined as required through the thesis; there are in particular clusters of definitions in subsection 1.3.1 (note especially the last paragraph on p. 24), the introduction to Chapter 2, and Section 3.1.

A further point may be useful to note: when $n \in \mathrm{Z}^+$, expressions such as α^n or $(\eta\underline{\pi}\underline{\gamma})^n$ show that the symbol, or string of symbols enclosed by parentheses, is repeated so as to occur n times. There is no implication of raising to a power, in the usual sense, except in a context in which juxtaposition of symbols denotes multiplication. In Sections 1.1, 1.2, and most of 1.3, juxtaposition does not denote multiplication.

CHAPTER 1

WORDS

In Philip Hall's well-known "contribution to the theory of groups of prime-power order" [9] it is established that in an arbitrary group there is a relationship between the operations of commutation, raising to the pth power, and multiplication. In particular, his Theorem 3.2, as modified in 1.6.1 of this thesis, says that for arbitrary elements α and β of a group, prime p , and positive integer h ,

$$(\alpha\beta)^{p^h} = \alpha^{p^h}\beta^{p^h}\prod\left\{\kappa_g^{p^{h(g)}} : g \in \Gamma\right\}$$

where for g in Γ , $h(g)$ is an integer satisfying $0 \le h(g) \le h$ and κ_g is a commutator with at least $\max\{2, p^{h-h(g)}\}$ entries from the set $\{\alpha, \beta\}$.

In this chapter, this and other relationships between these operations are studied. In the spirit of the comment made by Martin Ward at the end of the introduction to [29] (p. 346), this study is carried out by means of a free universal algebra, with operations $\underline{\gamma}$, $\underline{\pi}$ and $\underline{\mu}$ corresponding with commutation, raising to the pth power, and multiplication in a group. Varieties of $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras in which certain types of laws hold are defined. These earlier sections of the chapter culminate in Theorem 1.5.7, which says that every such variety has a law linking an arbitrary word in the $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-word algebra with one in which all operations $\underline{\gamma}$ precede all operations $\underline{\pi}$ which in turn precede all operations $\underline{\mu}$ (in group-theoretical terms, a product of p-power powers of commutators) satisfying certain weight relationships.

Section 1.6 shows that the variety of all groups satisfies laws of the appropriate types, so that the earlier results apply to all groups. In Section 1.7, certain descending central series of groups are defined and some of their properties are listed. Groups in which these series reach the trivial subgroup in finitely many steps are briefly discussed.

1.1 A $\{\underline{\gamma}, \underline{\pi}\}$-word algebra

1.1.1 WORDS AND SUBWORDS

Let Ξ be a countably infinite set, and let A be the $\{\underline{\gamma}, \underline{\pi}\}$-word algebra on Ξ, where $\underline{\gamma}$ is a binary and $\underline{\pi}$ a unary operation (see, for example, Cohn [6], III.2, p. 117). The elements of A are of precisely three possible types:

(i) elements of Ξ;

(ii) $\{\underline{\gamma}, \underline{\pi}\}$-rows of the form $\alpha\beta\underline{\gamma}$ where α and β are words; and

(iii) $\{\underline{\gamma}, \underline{\pi}\}$-rows of the form $\alpha\underline{\pi}$ where α is a word.

Correspondingly, given a $\{\underline{\gamma}, \underline{\pi}\}$-word ψ in A, define φ to be a subword of ψ (written "$\varphi \le \psi$") if and only if either

(i) $\varphi = \psi$; or

(ii) $\psi = \alpha\beta\underline{\gamma}$ and either $\varphi \le \alpha$ or $\varphi \le \beta$; or

(iii) $\psi = \alpha\underline{\pi}$ and $\varphi \le \alpha$.

If either condition (ii) or condition (iii) holds, then φ is a proper subword of ψ (written "$\varphi < \psi$").

If φ is a proper subword of ψ, then there exists a finite sequence $\{\varphi_i : i \in \underline{n+1}\}$ of subwords of ψ such that

$$\varphi = \varphi_0 < \varphi_1 < \dots < \varphi_n = \psi$$

which is maximal in the sense that for i in $\underline{n}$, there is no subword φ' of ψ satisfying

$$\varphi_i < \varphi' < \varphi_{i+1} .$$

Corresponding to each word ψ in A there exists a finite subset, say $\{\xi_j : j \in \underline{m}\}$, of Ξ such that every element of Ξ which is equal (as a word) to a subword of ψ is contained in $\{\xi_j : j \in \underline{m}\}$. To emphasise this, the notation $\psi = \psi(\xi_0, \ldots, \xi_{m-1})$ will sometimes be used. There is no implication that every element of $\{\xi_j : j \in \underline{m}\}$ should occur as a subword of $\psi(\xi_0, \ldots, \xi_{m-1})$.

If $\underline{\alpha}$ is a homomorphism from A to a $\{\underline{\gamma}, \underline{\pi}\}$-algebra D such that for i in $\underline{m}$, $\xi_i\underline{\alpha} = \rho_i$, then $\psi\underline{\alpha}$ is denoted

$$\psi(\rho_1, \ldots, \rho_{m-1}) .$$

A subword of a word ψ which belongs to Ξ is called an initial subword of ψ .

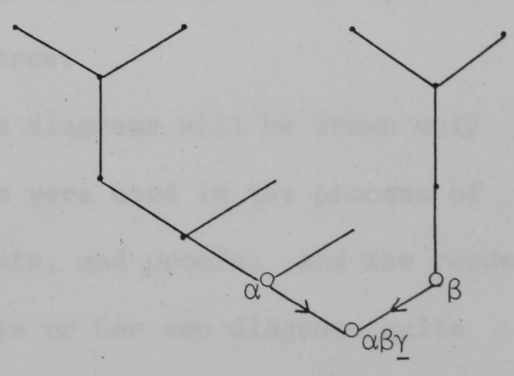
1.1.2 THE TREE OF A $\{\underline{\gamma}, \underline{\pi}\}$-WORD

The structure of a $\{\underline{\gamma}, \underline{\pi}\}$-word may conveniently be visualised in terms of a graph which is in fact a rooted tree, or arborescence, except that the directions of the arrows constituting its arcs are here reversed (for example, see C. Berge [4], Chapter 3, §3, p. 33). There is a one-one correspondence between vertices of the tree and subwords of the word, the root corresponding to the word itself. Every vertex is considered labelled with the corresponding subword. To each symbol $\underline{\gamma}$ in the word corresponds a pair of arcs directed toward the same vertex, and to each symbol $\underline{\pi}$ a single arc.

A formal definition may be made inductively, as follows:

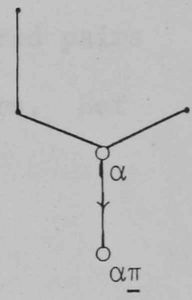
(i) The tree representing an element ξ of Ξ consists of a single vertex labelled ξ . $\circ\xi$

(ii) The tree representing a word $\alpha\beta\underline{\gamma}$ is obtained from the disjoint union of the trees representing α and β , drawn with that representing α on the left, by adjoining a new vertex labelled $\alpha\beta\underline{\gamma}$ which becomes its root, and an arc



directed toward this new vertex from each of the previous roots labelled α and β . The new arcs are referred to as $\underline{\gamma}$-arcs, sometimes as left- and right-$\underline{\gamma}$-arcs, respectively.

(iii) The tree representing a word $\alpha\underline{\pi}$ is obtained from the tree representing α by adjoining a new vertex labelled $\alpha\underline{\pi}$ which becomes its root, and an arc directed toward it from the previous root labelled α . The new arc is called a $\underline{\pi}$-arc.

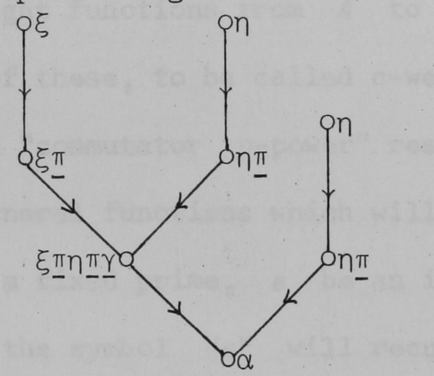


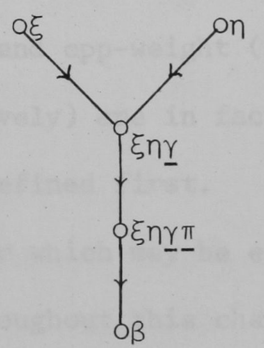
The introduction of a distinction between left- and right-$\underline{\gamma}$-arcs directed toward a vertex makes it possible to establish a one-one correspondence between such trees with labelled vertices and elements of A .

As examples, if ξ and η are elements of Ξ , and α and β are the words

$$\alpha = \xi\underline{\pi}\eta\underline{\pi}\underline{\gamma}\eta\underline{\pi}\underline{\gamma} \quad\text{and}\quad \beta = \xi\eta\underline{\gamma}\underline{\pi}\underline{\pi} ,$$

then the trees representing α and β are:





When distinct subwords of a given word are equal as words, then different vertices of the tree of that given word have the same label.

In the tree corresponding to a word ψ , a vertex which is not the terminal point of an arc must correspond to an initial subword of ψ . Such a vertex is called an initial vertex of the tree.

Through the remainder of Chapter 1, tree diagrams will be drawn only occasionally. However, large numbers of them were used in the process of formulating the various definitions, statements, and proofs; and the reader will probably find it an advantage to draw his or her own diagrams quite frequently, and to formulate definitions in terms of the diagrams.

1.1.3 INTEGERS ASSOCIATED WITH $\{\underline{\gamma}, \underline{\pi}\}$-WORDS

The first two functions to be defined are from the set of ordered pairs $\{(\varphi, \psi) : \psi \in A$ and $\varphi \leq \psi\}$ to the set N of non-negative integers. Let ψ be a word in A and φ a subword of ψ, and let

$$\varphi = \varphi_0 < \varphi_1 < \ldots < \varphi_n = \psi$$

be the maximal sequence of subwords linking φ with ψ described earlier. Define $k(\varphi, \psi)$ and $l(\varphi, \psi)$ to be the number of subwords in the set $\{\varphi_i : 1 \leq i \leq n\}$ which terminate in $\underline{\gamma}$ and in $\underline{\pi}$, respectively. (Note that $\varphi_0 = \varphi$ is *not* contained in the set.)

In terms of the tree representing ψ, $k(\varphi, \psi)$ is the number of $\underline{\gamma}$-arcs and $l(\varphi, \psi)$ the number of $\underline{\pi}$-arcs in the directed path from φ to ψ.

Several weight functions from A to Z^+ will now be introduced. The most important of these, to be called c-weight and cpp-weight (standing for "commutator" and "commutator p-power" respectively) are in fact special cases of more general functions which will be defined first.

Let p be a fixed prime, e be an integer which may be either 1 or p (this use of the symbol "e" will recur throughout this chapter, and in Chapters 3 and 4) and a and b integers satisfying the conditions that $a \geq b \geq 0$ and $a \geq 1$. The function $w^e_{a,b} : A \to \mathrm{Z}^+$ is defined inductively as follows:

(i) if $\psi \in \Xi$, then $w^e_{a,b}(\psi) = a$;

(ii) if $\psi = \alpha\beta\underline{\gamma}$, then $w^e_{a,b}(\psi) = w^e_{a,b}(\alpha) + w^e_{a,b}(\beta)$; and

(iii) if $\psi = \alpha\underline{\pi}$, then $w^e_{a,b}(\psi) = e\, w^e_{a,b}(\alpha) + d(1, e)b$ where

$$d(1, e) = \begin{cases} 1 & \text{if } e = 1 , \\ 0 & \text{if } e \neq 1 . \end{cases}$$

For example, if $p = 5$ and ξ and η are elements of Ξ , then

$$w^{1}_{a,b}(\xi\eta\underline{\gamma}\eta\underline{\gamma}) = 3a ,$$

$$w^{p}_{a,b}(\xi\eta\underline{\gamma}\eta\underline{\gamma}) = 3a ,$$

$$w^{1}_{a,b}(\xi\underline{\pi}\eta\underline{\gamma}\eta\underline{\gamma}\underline{\pi}\underline{\pi}) = 3a + 3b ,$$

and

$$w^{p}_{a,b}(\xi\underline{\pi}\eta\underline{\gamma}\eta\underline{\gamma}\underline{\pi}\underline{\pi}) = 175a .$$

The function $w^{p}_{a,b}$ is independent of b ; in fact it is easy to see that for all ψ in A ,

$$w^{p}_{a,b}(\psi) = a\, w^{p}_{1,0}(\psi) .$$

The function $w^{1}_{a,b}$ does depend on b . An easy way to compute its value is to note that if the word ψ contains l symbols $\underline{\gamma}$ and m symbols $\underline{\pi}$, then $w^{1}_{a,b}(\psi) = a(l+1) + bm$.

In Chapter 3, these weight functions will be used in a normal subgroup of finite p-power index in a group, and the integers a and b depend on the quotient group. When the quotient group is trivial, $a = 1$ and $b = 0$. This special case is the important one already referred to: $w^{1}_{1,0}$ is the c-weight function and $w^{p}_{1,0}$ the cpp-weight function.

These simpler functions may be defined for a word, not only as a whole, but also with respect to each element of the generating set Ξ . That is, functions from $A \times \Xi$ to $\mathbb{N}$ are defined as follows:

for all ξ in Ξ ,

(i) if $\psi \in \Xi$, then

$$\text{c-wt}(\psi,\ \xi) = \text{cpp-wt}(\psi,\ \xi) = \begin{cases} 1 & \text{if } \psi = \xi\ , \\ 0 & \text{if } \psi \neq \xi\ ; \end{cases}$$

(ii) if $\psi = \alpha\beta\underline{\gamma}$, then

$$\text{c-wt}(\psi,\ \xi) = \text{c-wt}(\alpha,\ \xi) + \text{c-wt}(\beta,\ \xi)$$

and

$$\text{cpp-wt}(\psi,\ \xi) = \text{cpp-wt}(\alpha,\ \xi) + \text{cpp-wt}(\beta,\ \xi)\ ;$$

and

(iii) if $\psi = \alpha\underline{\pi}$, then

$$\text{c-wt}(\psi,\ \xi) = \text{c-wt}(\alpha,\ \xi)$$

and

$$\text{cpp-wt}(\psi,\ \xi) = p.\text{cpp-wt}(\alpha,\ \xi)\ .$$

It is easy to see that c-wt$(\psi,\ \xi)$ is equal to the number of occurrences of the symbol ξ in the word ψ . This in turn is equal to the number of vertices labelled ξ in the tree representing ψ .

Correspondingly,

$$\text{cpp-wt}(\psi,\ \xi) = \sum \{p^{l(\rho,\psi)} : \rho \leq \psi \text{ and } \rho = \xi\}\ .$$

Also, for each ψ in A ,

$$\text{c-wt}(\psi) = w^1_{1,0}(\psi) = \sum \{\text{c-wt}(\psi,\ \xi) : \xi \in \Xi\}$$

and

$$\text{cpp-wt}(\psi) = w^p_{1,0}(\psi) = \sum \{\text{cpp-wt}(\psi,\ \xi) : \xi \in \Xi\}\ .$$

1.1.4 PRE-ORDERS ON A

A relation denoted $\leq'$ is defined on A by the condition that $\alpha \leq' \beta$ if and only if every weight function defined in 1.1.3 takes at α a value less than or equal to its value at β . More formally, $\alpha \leq' \beta$ if and only if for all integers a and b such that $a \geq b \geq 0$ and $a \geq 1$, all e in $\{1,\ p\}$, and all ξ in Ξ ,

$$w^e_{a,b}(\alpha) \leq w^e_{a,b}(\beta)\ ,$$

$$\text{c-wt}(\alpha, \xi) \leq \text{c-wt}(\beta, \xi) \ ,$$

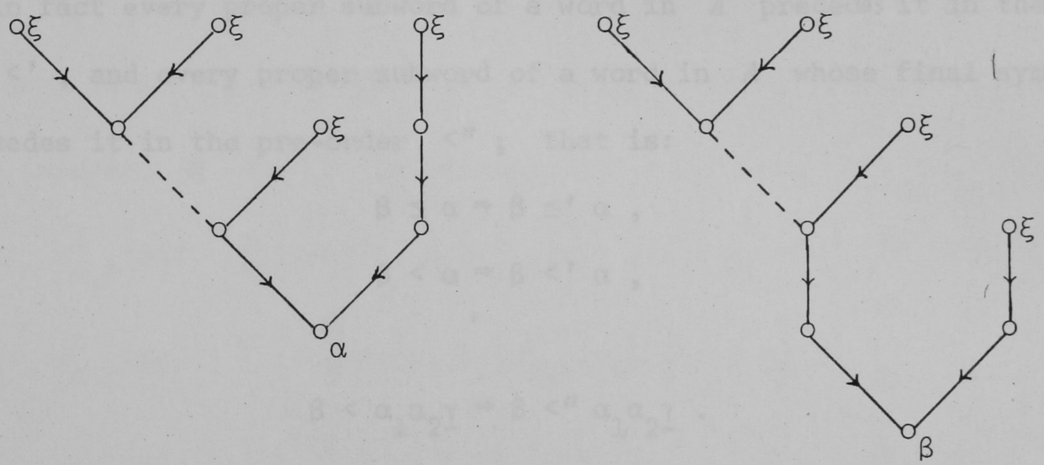
and

$$\text{cpp-wt}(\alpha, \xi) \leq \text{cpp-wt}(\beta, \xi) \ .$$

The relation $\leq'$ is clearly transitive, and is therefore a pre-order. It is reflexive, but is not antisymmetric (even on equivalence classes obtained by allowing commutative and associative rearrangements of the arguments of $\underline{\gamma}$).

For example, if

$$\alpha = \xi(\xi\underline{\gamma})^{p-1}\xi\underline{\pi}\,\underline{\pi}\,\underline{\gamma} \quad \text{and} \quad \beta = \xi(\xi\underline{\gamma})^{p-1}\underline{\pi}\xi\underline{\pi}\,\underline{\gamma} \ ,$$



then

$$w^{1}_{a,b}(\alpha) = \quad w^{1}_{a,b}(\beta) = a(p+1) + 2b \ ,$$

$$w^{p}_{a,b}(\alpha) = \quad w^{p}_{a,b}(\beta) = a\left(p^2+p\right) \ ,$$

$$\text{c-wt}(\alpha, \xi) = \quad \text{c-wt}(\beta, \xi) = p + 1 \ ,$$

and

$$\text{cpp-wt}(\alpha, \xi) = \text{cpp-wt}(\beta, \xi) = p^2 + p \ .$$

Nevertheless the structures of the words α and β are quite different, as can be seen by considering values of $l(\rho, \alpha)$ and $l(\rho, \beta)$ for various initial subwords ρ .

Non-reflexive relations $<'$ and $<''$ are now defined as follows:

$\alpha <' \beta$ if and only if $\alpha \leq' \beta$ and for all integers a and b such that $a \geq b \geq 0$ and $a \geq 1$,

$$w^1_{a,b}(\alpha) < w^1_{a,b}(\beta) .$$

$\alpha <'' \beta$ if and only if $\alpha <' \beta$ and for all integers a and b such that $a \geq b \geq 0$ and $a \geq 1$,

$$w^p_{a,b}(\alpha) < w^p_{a,b}(\beta) .$$

Note that $<'$ and $<''$ are also pre-orders. The preceding example makes clear that "$\alpha <' \beta$" is a stronger statement than "$\alpha \leq' \beta$ and $\alpha \neq \beta$".

Every subword of an arbitrary word in A precedes it in the pre-order $\leq'$. In fact every proper subword of a word in A precedes it in the pre-order $<'$, and every proper subword of a word in A whose final symbol is $\underline{\gamma}$ precedes it in the pre-order $<''$; that is:

$$\beta \leq \alpha \Rightarrow \beta \leq' \alpha ,$$

$$\beta < \alpha \Rightarrow \beta <' \alpha ,$$

and

$$\beta < \alpha_1\alpha_2\underline{\gamma} \Rightarrow \beta <'' \alpha_1\alpha_2\underline{\gamma} .$$

The pre-orders are preserved under the operations $\underline{\gamma}$ and $\underline{\pi}$ in the sense that

$$\alpha \leq' \varphi,\ \beta \leq' \psi \Rightarrow \alpha\beta\underline{\gamma} \leq' \varphi\psi\underline{\gamma}$$

and

$$\alpha \leq' \varphi \Rightarrow \alpha\underline{\pi} \leq' \varphi\underline{\pi} ;$$

and if one of the symbols $\leq'$ on the left of one of these implications is replaced by the symbol $<'$ or $<''$, then that on the right may be replaced by the same replacement symbol.

Further properties of note are that

$$\alpha\beta\underline{\gamma} \leq' \beta\alpha\underline{\gamma} ,$$

$$\alpha\beta\delta\underline{\gamma}\underline{\gamma} \leq' \alpha\beta\underline{\gamma}\delta\underline{\gamma} \leq' \alpha\beta\delta\underline{\gamma}\underline{\gamma} ,$$

and except in the case $a = b$, $p = 2$, and $w^1_{a,a}(\alpha) = a$,

$$\alpha\underline{\pi} <' \alpha(\alpha\underline{\gamma})^{p-1} .$$

* * *

The following result and its corollary deal with the behaviour of the weight functions and pre-order relations under operations that may be regarded either as endomorphisms of A or as "substitutions" into words.

1.1.5 LEMMA. *Let* $\varphi = \varphi(\xi_0, \ldots, \xi_m)$ *be a word in* A *, and for* i *in* $\underline{m}$ *let* $\alpha_i = \alpha_i(\xi_0, \ldots, \xi_{l(i)-1})$ *also be words in* A *. (Without loss of generality, it may be assumed that* $m \geq l(i)$ *for all* i *in* $\underline{m}$ *; then set* $\alpha_i = \alpha_i(\xi_0, \ldots, \xi_m)$ *for all* i *in* $\underline{m}$ *.) Let* $\underline{\alpha}$ *be an endomorphism of* A *such that* $\xi_i\underline{\alpha} = \alpha_i$ *for all* i *in* $\underline{m}$ *. Then, for all* h *in* $\underline{m}$ *and all integers* a *and* b *satisfying* $a \geq b \geq 0$ *and* $a \geq 1$ *,*

(a) $w^1_{a,b}(\varphi\underline{\alpha}) = \sum\left\{\text{c-wt}(\varphi, \xi_i)w^1_{a,b}(\alpha_i) : i \in \underline{m}\right\} + w^1_{a,b}(\varphi) - w^1_{a,0}(\varphi)$,

(b) $w^p_{a,b}(\varphi\underline{\alpha}) = \sum\left\{\text{cpp-wt}(\varphi, \xi_i)w^p_{a,b}(\alpha_i) : i \in \underline{m}\right\}$,

(c) $\text{c-wt}(\varphi\underline{\alpha}, \xi_h) = \sum\{\text{c-wt}(\varphi, \xi_i)\text{c-wt}(\alpha_i, \xi_h) : i \in \underline{m}\}$ *, and*

(d) $\text{cpp-wt}(\varphi\underline{\alpha}, \xi_h) = \sum\{\text{cpp-wt}(\varphi, \xi_i)\text{cpp-wt}(\alpha_i, \xi_h) : i \in \underline{m}\}$.

Proof. The proofs of *(a)*, *(b)*, *(c)* and *(d)* are very similar in outline, proceeding by induction on the number of symbols $\underline{\gamma}$ or $\underline{\pi}$ in the word φ . Since the form of the result *(a)* is slightly more complicated, the proof of *(a)* is given here, and the other proofs are omitted.

Three cases are considered:

(i) If φ contains no symbols $\underline{\gamma}$ or $\underline{\pi}$, then without loss of generality $\varphi = \xi_0$ and $\varphi\underline{\alpha} = \alpha_0$, whence

$$w^1_{a,b}(\varphi\underline{\alpha}) = w^1_{a,b}(\alpha_0) + a - a ,$$

as required.

For cases (ii) and (iii), in each of which φ contains at least one symbol $\underline{\gamma}$ or $\underline{\pi}$, suppose inductively that the result is established for all words with fewer such symbols.

(ii) If $\varphi = \psi_1\psi_2\underline{\gamma}$, whence $\varphi\underline{\alpha} = \psi_1\underline{\alpha}\psi_2\underline{\alpha}\underline{\gamma}$, then

$$\begin{aligned} w^1_{a,b}(\varphi\underline{\alpha}) &= w^1_{a,b}\left(\psi_1\underline{\alpha}\right) + w^1_{a,b}\left(\psi_2\underline{\alpha}\right) \\ &= \sum\left\{\text{c-wt}\left(\psi_1,\ \xi_i\right)w^1_{a,b}\left(\alpha_i\right)\ :\ i \in \underline{m}\right\} + w^1_{a,b}\left(\psi_1\right) - w^1_{a,0}\left(\psi_1\right) + \\ &\qquad + \sum\left\{\text{c-wt}\left(\psi_2,\ \xi_i\right)w^1_{a,b}\left(\alpha_i\right)\ :\ i \in \underline{m}\right\} + w^1_{a,b}\left(\psi_2\right) - w^1_{a,0}\left(\psi_2\right) \\ &= \sum\left\{\text{c-wt}\left(\varphi,\ \xi_i\right)w^1_{a,b}\left(\alpha_i\right)\ :\ i \in \underline{m}\right\} + w^1_{a,b}(\varphi) - w^1_{a,0}(\varphi)\ , \end{aligned}$$

as required. □

(iii) If $\varphi = \psi\underline{\pi}$, whence $\varphi\underline{\alpha} = \psi\underline{\alpha}\underline{\pi}$, then

$$\begin{aligned} w^1_{a,b}(\varphi\underline{\alpha}) &= w^1_{a,b}(\psi\underline{\alpha}) + b \\ &= \sum\left\{\text{c-wt}\left(\psi,\ \xi_i\right)w^1_{a,b}\left(\alpha_i\right)\ :\ i \in \underline{m}\right\} + w^1_{a,b}(\psi) - w^1_{a,0}(\psi) + b \\ &= \sum\left\{\text{c-wt}\left(\varphi,\ \xi_i\right)w^1_{a,b}\left(\alpha_i\right)\ :\ i \in \underline{m}\right\} + w^1_{a,b}(\varphi) - w^1_{a,0}(\varphi)\ , \end{aligned}$$

as required.

1.1.6 COROLLARY. *Let* $\varphi = \varphi\left(\xi_0, \ldots, \xi_{m-1}\right)$ *be a word in* A *, and for* i *in* $\underline{m}$ *let* α_i *and* β_i *be words in* A *such that* $\alpha_i \le' \beta_i$ *. Then*

$$\varphi\left(\alpha_0, \ldots, \alpha_{m-1}\right) \le' \varphi\left(\beta_0, \ldots, \beta_{m-1}\right) .$$

If, further, there exists i *in* $\underline{m}$ *such that* $\text{c-wt}\left(\varphi, \xi_i\right) \ge 1$ *and* $\alpha_i <' \beta_i$ *(respectively* $<''$ *) then* $\varphi\left(\alpha_0, \ldots, \alpha_{m-1}\right) <' \varphi\left(\beta_0, \ldots, \beta_{m-1}\right)$ *(respectively* $<''$ *).*

Proof. The result follows immediately from Lemma 1.1.5 and the definitions.

1.2 Some $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras

1.2.1 A $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-WORD ALGEBRA

Let Ξ be a countably infinite set, as before, and let B be the $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-word algebra on Ξ where $\underline{\gamma}$ and $\underline{\mu}$ are binary operations and $\underline{\pi}$ is a unary operation. For the rest of this chapter, indeed in later chapters as well, references to "words" without further qualification are to $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-words which are elements of B.

Subwords of words in B are defined in the same way as subwords of $\{\underline{\gamma}, \underline{\pi}\}$-words in A, except that there are four types of word, instead of three, to be considered.

Words in B which contain neither symbol $\underline{\pi}$ nor $\underline{\mu}$ are called c-words. Those which do not contain the symbol $\underline{\mu}$ are called cpp-words. (Again the reference is to "commutator" and "commutator p-power".) The cpp-words in B, together with the operations $\underline{\gamma}$ and $\underline{\pi}$, form a $\{\underline{\gamma}, \underline{\pi}\}$-word algebra. All definitions of notation, numerical quantities, and pre-order relations given in Section 1.1 are taken to apply unchanged to cpp-words in B. Words of the form $\alpha\underline{\pi}^n$ where α is a c-word and $n \geq 0$ will be called simple cpp-words, abbreviated to scpp-words.

Many of the definitions taken above to apply to cpp-words in B will now be extended to apply to the whole of B.

In a $\underline{\mu}$-product of cpp-words, the factor on which some weight function takes the least value is generally the most significant, at least as far as that weight function is concerned. Thus to the three-part inductive definitions already borrowed from 1.1.3 are added fourth parts, as follows:

(iv) If $\psi = \alpha\beta\underline{\mu}$, then for e in $\{1, p\}$ and ξ in Ξ,

$$w^e_{a,b}(\psi) = \min\left\{w^e_{a,b}(\alpha),\ w^e_{a,b}(\beta)\right\},$$

$$\text{c-wt}(\psi, \xi) = \min\{\text{c-wt}(\alpha, \xi),\ \text{c-wt}(\beta, \xi)\},$$

and

$$\text{cpp-wt}(\psi, \xi) = \min\{\text{cpp-wt}(\alpha, \xi),\ \text{cpp-wt}(\beta, \xi)\}.$$

Together with the first three parts, this defines each weight function on the whole of B .

The maximal sequence of subwords linking a given subword with the word containing it is defined exactly as before, as are the integers $k(\varphi, \psi)$ and $l(\varphi, \psi)$. Subwords terminating in $\underline{\mu}$ are simply ignored when the counts required by definitions of $k(\varphi, \psi)$ and $l(\varphi, \psi)$ are carried out.

In order to simplify later notation, mappings $\underline{\gamma}'$ and $\underline{\mu}'$ from the set of finite, non-empty ordered sets of elements in B to B are defined inductively as follows:

Let Δ be a finite, non-empty, ordered set of elements in B . If $|\Delta| = 1$, say $\Delta = \{\alpha\}$, then

$$\Delta\underline{\mu}' = \alpha \quad \text{and} \quad \Delta\underline{\gamma}' = \alpha \ .$$

If $|\Delta| > 1$, then let Δ^* be the ordered set obtained by deleting the "last element", α say, of Δ ; and define

$$\Delta\underline{\mu}' = \Delta^*\underline{\mu}'\alpha\underline{\mu} \quad \text{and} \quad \Delta\underline{\gamma}' = \Delta^*\underline{\gamma}'\alpha\underline{\gamma} \ .$$

That is, the effect of $\underline{\mu}'$ or $\underline{\gamma}'$ on an ordered set is that of successive application of the operation $\underline{\mu}$ or $\underline{\gamma}$ respectively to the elements of the set arranged with left-normed bracketting.

Analogues of the weight functions used in 1.1.4 to define relations $\leq'$, $<'$ and $<''$ on A have already been defined on B . Corresponding relations, again denoted $\leq'$, $<'$ and $<''$ are now defined on the whole of B by using the same definitions used in 1.1.4. The new relations have all the properties described in 1.1.4, and are preserved in the same way by the operations $\underline{\gamma}$ and $\underline{\pi}$. The following further properties associated with the operation $\underline{\mu}$ are also easily checked:

(a) $\alpha \leq' \alpha\alpha\underline{\mu}$,

(b) $\alpha_1\alpha_2\underline{\mu} \leq' \alpha_2\alpha_1\underline{\mu}$,

(c) $\alpha_1\alpha_2\underline{\mu} \leq' \alpha_1$, and

(d) $\alpha_1 \leq' \beta_1$ and $\alpha_2 \leq' \beta_2 \Rightarrow \alpha_1\alpha_2\underline{\mu} \leq' \beta_1\beta_2\underline{\mu}$

(and implications similar to (d) involving $<'$ and $<''$). From the above properties may be deduced:

(e) $\alpha \le' \beta_1$ and $\alpha \le' \beta_2 \Leftrightarrow \alpha \le \beta_1\beta_2\underline{\mu}$.

One special situation in which these relations will frequently be used deserves comment. If Γ and Δ are finite (non-empty) ordered sets of cpp-words, then the relation $\Gamma\underline{\mu}' \le' \Delta\underline{\mu}'$ holds if and only if for each weight function w defined earlier and each element β in Δ , there exists α in Γ such that $w(\alpha) \le w(\beta)$. Since in general the choice of α depends on w as well as on β , it does not necessarily follow that for each β in Δ there exists α in Γ such that $\alpha \le' \beta$. A useful exception holds when $|\Gamma| = 1$; the statements "$\alpha \le' \Delta\underline{\mu}'$" and "$\alpha \le' \beta$ for all β in Δ" are equivalent.

A direct parallel to Lemma 1.1.5 does not hold in B . Nevertheless, a method of proof similar to that for 1.1.5 gives an analogue of Corollary 1.1.6.

1.2.2 LEMMA. *Let* $\varphi = \varphi(\xi_0, \ldots, \xi_{m-1})$ *be a word (in* B *) and for* i *in* $\underline{m}$ *let* α_i *and* β_i *be words such that* $\alpha_i \le' \beta_i$. *Then*

$$\varphi(\alpha_0, \ldots, \alpha_{m-1}) \le' \varphi(\beta_0, \ldots, \beta_{m-1}) .$$

If, further, for some i *in* $\underline{m}$ *,*

$$\text{c-wt}(\varphi, \xi_i) \ge 1 \text{ and } \alpha_i <' \beta_i \text{ (respectively } <'' \text{)},$$

then

$$\varphi(\alpha_0, \ldots, \alpha_{m-1}) <' \varphi(\beta_0, \ldots, \beta_{m-1}) \text{ (respectively } <'' \text{)}.$$

Proof. (i) If φ contains no symbols $\underline{\gamma}$, $\underline{\pi}$, or $\underline{\mu}$, then without loss of generality $\alpha = \xi_0$, and all conclusions of the lemma clearly hold.

For the remaining cases in which φ does contain at least one symbol $\underline{\gamma}$, $\underline{\pi}$, or $\underline{\mu}$, suppose inductively that the result is already established for all words with fewer such symbols. Let $\underline{\alpha}$ and $\underline{\beta}$ be endormophisms of B defined by

$$\xi_{\underline{i}}\underline{\alpha} = \alpha_i \text{ and } \xi_{\underline{i}}\underline{\beta} = \beta_i \text{ for all } i \text{ in } \underline{m},$$

and

$$\xi\underline{\alpha} = \xi_0 \text{ and } \xi\underline{\beta} = \xi_0 \text{ for all } \xi \text{ in } \Xi\backslash\{\xi_i : i \in \underline{m}\}.$$

Hence $\varphi(\alpha_0, \ldots, \alpha_{m-1}) = \varphi\underline{\alpha}$ and $\varphi(\beta_0, \ldots, \beta_{m-1}) = \varphi\underline{\beta}$.

(ii) If $\varphi = \psi_1\psi_2\underline{\gamma}$, then by the inductive hypothesis $\psi_1\underline{\alpha} \leq' \psi_1\underline{\beta}$ and $\psi_2\underline{\alpha} \leq' \psi_2\underline{\beta}$. Hence, since the pre-order is preserved by the operation $\underline{\gamma}$,

$$\varphi\underline{\alpha} = (\psi_1\psi_2\underline{\gamma})\underline{\alpha} \leq' (\psi_1\psi_2\underline{\gamma})\underline{\beta} = \varphi\underline{\beta},$$

as required. If for some i in $\underline{m}$, c-wt$(\varphi, \xi_i) \geq 1$ and $\alpha_i <' \beta_i$ (respectively $\alpha_i <'' \beta_i$), then there is an element d in $\{1, 2\}$ such that c-wt$(\psi_d, \xi_i) \geq 1$, whence by induction $\psi_d\underline{\alpha} <' \psi_d\underline{\beta}$ (respectively $<''$); and again the required conclusion follows.

(iii) If $\varphi = \psi\underline{\pi}$, then from any of the relations $\psi\underline{\alpha} \leq' \psi\underline{\beta}$, $\psi\underline{\alpha} <' \psi\underline{\beta}$, or $\psi\underline{\alpha} <'' \psi\underline{\beta}$, the corresponding relation between $\varphi\underline{\alpha}$ and $\varphi\underline{\beta}$ follows immediately.

(iv) If $\varphi = \psi_1\psi_2\underline{\mu}$, then the inductive hypothesis states that $\psi_1\underline{\alpha} \leq' \psi_1\underline{\beta}$ and $\psi_2\underline{\alpha} \leq' \psi_2\underline{\beta}$, whence by property (d) in 1.2.1, the required result follows. If for some i in $\underline{m}$, c-wt$(\varphi, \xi_i) \geq 1$, then both c-wt$(\psi_1, \xi_i) \geq 1$ and c-wt$(\psi_2, \xi_i) \geq 1$. Thus, inductively, if $\alpha_i <' \beta_i$ (respectively $\alpha_i <'' \beta_i$) then both $\psi_1\underline{\alpha} <' \psi_1\underline{\beta}$ and $\psi_2\underline{\alpha} <' \psi_2\underline{\beta}$ (respectively $<''$ in both relations) and the required result follows as before.

1.2.3 OTHER $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-ALGEBRAS

Every countable $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra is a homomorphic image of the word algebra B described in 1.2.1. The homomorphic images of c-words, cpp-words, and scpp-words are referred to as c-elements, cpp-elements, and scpp-elements respectively. In a group, a c-element is more usually called a commutator, and the homomorphic images of the initial subwords of the corresponding c-word are called its entries.

There are difficulties in extending definitions of weight functions, or the pre-order relations $\leq'$, $<'$ and $<''$ to general $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras. For example, there may be many different words in B, with different weights, mapped by a homomorphism to the same image.

For a given weight function w on B, a countable $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra D, and a surjective homomorphism $\underline{\alpha} : B \to D$, it is possible to define a function from D to $Z^+ \cup \{\infty\}$ by setting, for all δ in D,

$$w(\delta, \underline{\alpha}) = \begin{cases} \max\{w(\varphi) : \varphi\underline{\alpha} = \delta\} & \text{if this exists} \\ \infty & \text{if no such max exists.} \end{cases}$$

Correspondingly, a weight function from D to $Z^+ \cup \{\infty\}$ which is independent of a particular homomorphism may be defined by setting, for all δ in D,

$$w(\delta) = \begin{cases} \max\{w(\delta, \underline{\alpha}) : \underline{\alpha} \text{ a surjective homomorphism from } B \text{ to } D\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if such a maximum exists} \\ \infty \qquad\qquad\qquad\qquad\qquad\qquad \text{if no such maximum exists.} \end{cases}$$

This procedure involves the possibility that different weight functions might, for a fixed element of D, take values related to quite differently-structured elements of B, so that the relationship between values taken by different weight functions on a fixed element of D is lost. Such an approach is implicit in the use, in later chapters, of the weight ideals defined in the next subsection.
However, weight functions as such will not be defined on algebras other than the word-algebras A and B.

In this chapter, if δ is an element of an arbitrary $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra D, and w is a weight function defined on B, then the sentence

$$\text{"}w(\delta) = l\text{"}$$

will be defined to have the weak meaning "there exist φ in B and a surjective homomorphism $\underline{\alpha} : B \to D$ such that $w(\varphi) = l$ and $\varphi\underline{\alpha} = \delta$ ". This is clearly of no use for making comparisons between elements of D, as for arbitrary δ in D the statement "$w^e_{a,b}(\delta) = a$" is true.

Comparisons between elements of D may be made in terms of relations, again denoted $\leq'$, $<'$ and $<''$, defined as follows:

for arbitrary ρ and σ in D, $\rho \leq' \sigma$ if and only if for every word φ in B and surjective homomorphism $\underline{\alpha}^*$ from B to D such that $\varphi\underline{\alpha}^* = \rho$, there exist an element ψ in B and surjective homomorphism $\underline{\alpha}$ from B to D such that $\varphi \leq' \psi$, $\psi\underline{\alpha} = \sigma$, and for all ξ in Ξ which occur as subwords of φ, $\xi\underline{\alpha} = \xi\underline{\alpha}^*$. (The last condition implies that $\varphi\underline{\alpha} = \varphi\underline{\alpha}^* = \rho$.)

Relations $<'$ and $<''$ are defined similarly. All three relations defined in this way are transitive; the proof is routine, and is omitted.

1.2.4 WEIGHT IDEALS OF $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-ALGEBRAS

As before, let p be a fixed prime, and a and b integers such that $a \geq b \geq 0$ and $a \geq 1$. Suppose that the generating set Ξ for B is $\Xi = \{\xi_i : i \in \mathbb{N}\}$. Now, corresponding to sets of non-negative integers

$$\{l(e) : e = 1 \text{ or } e = p\},\ \{w(i) : i \in \mathbb{N}\} \text{ and } \{v(i) : i \in \mathbb{N}\}$$

satisfying the conditions that $\sum\{w(i) : i \in \mathbb{N}\}$ and $\sum\{v(i) : i \in \mathbb{N}\}$ are finite, let I be the set of words α in B satisfying the conditions:

(a) $\forall\, e \in \{1, p\}$, $w^e_{a,b}(\alpha) \geq l(e)$,

(b) $\forall\, i \in \mathbb{N}$, $\text{c-wt}(\alpha, \xi_i) \geq w(i)$, and

(c) $\forall\, i \in \mathbb{N}$, $\text{cpp-wt}(\alpha, \xi_i) \geq v(i)$.

The set I is closed under the operations $\underline{\gamma}$, $\underline{\pi}$ and $\underline{\mu}$. In fact, if $\alpha \in I$ and $\beta \in B$, then $\alpha\beta\underline{\gamma} \in I$ and $\beta\alpha\underline{\gamma} \in I$. If the operation $\underline{\mu}$ is regarded as playing the role normally played in an algebra by addition, the set I may be called an *ideal* of B.

If D is an arbitrary $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra and $\underline{\alpha}$ is a surjective homomorphism from B to D, then the image in D of an ideal in B is again an ideal, in the same sense.

Conditions (b) and (c) may be made trivial by setting $w_i = v_i = 0$ for all i in N. The ideals defined in this way by condition (a) only are easily seen to be fully invariant, mapped into themselves by every endomorphism of B. Such ideals of B, and their images in other $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras, will be called *weight ideals*. Lemma 1.2.5 shows that a weight ideal in an arbitrary $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra is independent of the surjective homomorphism from B to the algebra used in its definition.

Of particular importance in groups are the weight ideals (fully invariant subgroups) defined as images of the ideals

$$\gamma_v(B) = \left\{\alpha \in B : w^1_{1,0}(\alpha) \geq v\right\}$$

and

$$\lambda_v(B) = \left\{\alpha \in B : w^p_{1,0}(\alpha) \geq v\right\}.$$

If the images in G of $\gamma_v(B)$ and $\lambda_v(B)$ are denoted $\gamma_v(G)$ and $\lambda_v(G)$ respectively, then the series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots \geq \gamma_v(G) \geq \ldots$$

is the lower central series of G, and the series

$$G = \lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_v(G) \geq \ldots$$

is the restricted elementary central series (cpp-series) of G mentioned in the introduction (p. 2). These and other series of weight ideals are discussed in more detail in Section 1.7.

1.2.5 LEMMA. *If I is a weight ideal of the $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-word algebra B, and if $\underline{\alpha}$ and $\underline{\beta}$ are surjective homomorphisms from B to a $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-word algebra D, then $I\underline{\alpha} = I\underline{\beta}$.*

Proof. From the symmetry of the situation, it is sufficient to show that $I\underline{\alpha} \subseteq I\underline{\beta}$. Let $\rho \in I\underline{\alpha}$. Then there exists a word, say

$\varphi = \varphi(\xi_0, \ldots, \xi_{z-1})$ in I such that $\varphi\underline{\alpha} = \rho$. Since $\underline{\beta}$ is a surjection, for each i in $\underline{z}$ there exists a word χ_i in B such that $\chi_i\underline{\beta} = \xi_i\underline{\alpha}$; then $\varphi(\chi_0, \ldots, \chi_{z-1})\underline{\beta} = \varphi\underline{\alpha} = \rho$. However, by Lemma 1.1.5 the word $\varphi(\chi_0, \ldots, \chi_{z-1})$ has at least as great a value under each weight function $w^e_{a,b}$ used in the definition of I as has $\varphi = \varphi(\xi_0, \ldots, \xi_{z-1})$; and hence $\varphi(\chi_0, \ldots, \chi_{z-1}) \in I$. Thus $\rho \in I\underline{\beta}$, as required. □

1.3 Group-like varieties of $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras

1.3.1 DEFINITIONS AND NOTATION

Following Cohn [6], IV.1, p. 162, define a *law* in a $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra to be a pair of words in $B \times B$. The law (θ, φ) is said to *hold* in a $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra D if under every homomorphism $\underline{\alpha} : B \to D$, the words θ and φ have the same image, that is, $\theta\underline{\alpha} = \varphi\underline{\alpha}$.

The *variety* defined by a set of laws is the class of algebras in which all laws of the set hold. The statement that a law (θ, φ) holds in a variety $\underline{\underline{V}}$ will be denoted $\theta \overset{\underline{V}}{=} \varphi$.

Let ξ, η, ζ, ξ_i for i in $\mathbb{N}$, and η_i for i in $\mathbb{N}$ be elements of Ξ . A *group-like variety* of $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras is defined to be a variety $\underline{\underline{V}}$ with laws of the form:

(i) $\xi\eta\underline{\mu}\zeta\underline{\mu} \overset{V}{=} \xi\eta\zeta\underline{\mu}\underline{\mu}$ (that is, the operation $\underline{\mu}$ is associative).

(ii) $\eta\xi\underline{\mu} \overset{V}{=} \xi\eta\underline{\mu}\eta\xi\underline{\gamma}\underline{\mu}$ (if $\underline{\mu}$ is identified as group multiplication, then this law identifies $\underline{\gamma}$ as the operation of commutation).

(iii) $\xi(\xi\underline{\mu})^{p-1} \overset{V}{=} \xi\underline{\pi}$ (again, if $\underline{\mu}$ is identified as group multiplication, then this law identifies $\underline{\pi}$ as the operation of raising to the pth power).

(iv) For arbitrary finite sub-ordered-sets Γ and Δ of $\mathbb{N}$,

$$\{\xi_g : g \in \Gamma\}\underline{\mu}'\{\eta_d : d \in \Delta\}\underline{\mu}'\underline{\gamma} \stackrel{\vee}{=} \{\xi_g\eta_d\underline{\gamma} : (g, d) \in \Gamma \times \Delta\}\underline{\mu}'\{\zeta_t : t \in \Theta_1\}\underline{\mu}'\underline{\mu}$$

where for t in Θ_1, ζ_t is a c-word and there exists a triple $(g(t), d(t), h(t))$ either belonging to $\Gamma \times \Delta \times \Gamma$ such that $g(t) \neq h(t)$ and

$$\xi_{g(t)}\eta_{d(t)}\underline{\gamma}\xi_{h(t)}\underline{\gamma} \leq' \zeta_t ,$$

or belonging to $\Gamma \times \Delta \times \Delta$ such that $d(t) \neq h(t)$ and

$$\xi_{g(t)}\eta_{d(t)}\underline{\gamma}\eta_{h(t)}\underline{\gamma} \leq' \zeta_t .$$

(If the product indexed by Θ_1 were deleted from the right-hand side, this law would say that the operation $\underline{\gamma}$ distributed over the operation $\underline{\mu}$.)

(v) For arbitrary finite sub-ordered-set Δ of $\mathbb{N}$, and m in $\mathbb{Z}^+$,

$$\{\xi_d : d \in \Delta\}\underline{\mu}'\underline{\pi}^m \stackrel{\vee}{=} \left\{\zeta_t\underline{\pi}^{l(t)} : t \in \Theta_2\right\}\underline{\mu}'$$

where for all t in Θ_2, ζ_t is a c-word and

$$\sum\{\text{c-wt}(\zeta_t, \xi_d) : d \in \Delta\} \geq p^{m-l(t)} .$$

Further, there is a subset Θ_2^* of Θ_2 such that

$$\left\{\zeta_t\underline{\pi}^{l(t)} : t \in \Theta_2^*\right\} = \left\{\xi_d\underline{\pi}^m : d \in \Delta\right\} ,$$

and for each t in $\Theta_2\backslash\Theta_2^*$, there are at least two distinct elements d in Δ such that $\text{c-wt}(\zeta_t, \xi_d) \geq 1$.

(If the terms indexed by elements of $\Theta_2\backslash\Theta_2^*$ were deleted from the right-hand side, this law would say that the operation $\underline{\pi}$ distributed over the operation $\underline{\mu}$.)

(vi) For arbitrary m in $\mathbb{Z}^+$,

(a) $\xi\underline{\pi}^m\eta\underline{\gamma} \stackrel{\vee}{=} \xi\eta\underline{\gamma}\underline{\pi}^m\left\{\zeta_t\underline{\pi}^{l(t)} : t \in \Theta_3\right\}\underline{\mu}'\underline{\mu}$, and

(b) $\xi\eta\underline{\pi}^m\underline{\gamma} \stackrel{\vee}{=} \left\{\zeta_t\underline{\pi}^{l(t)} : t \in \Theta_4\right\}\underline{\mu}'\xi\eta\underline{\gamma}\underline{\pi}^m\underline{\mu}$;

where for each t in Θ_3, ζ_t is a c-word such that

$\text{c-wt}(\zeta_t, \xi) \geq \max\{2, p^{m-l(t)}\}$ and $\text{c-wt}(\zeta_t, \eta) \geq 1$, and for each t in

Θ_4 , ζ_t is a c-word such that

$$\text{c-wt}(\zeta_t, \xi) \geq 1 \quad \text{and} \quad \text{c-wt}(\zeta_t, \eta) \geq \max\{2, p^{m-l(t)}\} .$$

(If all terms indexed by Θ_3 and Θ_4 were deleted from the right-hand sides, and if the language were stretched a little to refer in this way to a binary and a unary operation, then this law would say that the operations $\underline{\gamma}$ and $\underline{\pi}$ commuted.)

It should be noted that because elements of the sets $\{\zeta_t : t \in \Theta_i\}$ for $1 \leq i \leq 4$ are not fully specified, (iv), (v) and (vi) are not, strictly speaking, laws; but are conditions which laws must satisfy.

Note that in each of the six types of law, the expression on the right-hand side is a μ-product of scpp-words. Also, if the left-hand side is α and the right-hand side is β , it can be checked that $\alpha \leq' \beta$. If in types (iv), (v) and (vi), one of the terms whose omission from the right-hand side was discussed above as a simplification of the law is denoted δ , then $\alpha <' \delta$, provided that either $p \neq 2$ or $a > b$.

Note also that the statement "c-wt$(\zeta, \xi) \geq n$" is equivalent to the statement "the word ζ has at least n distinct subwords, all equal as words to ξ ". The latter interpretation will be used in some applications of these laws, particularly in the proof of Lemma 1.5.5.

Simplified notation and terminology is now introduced, in the light of the application of this theory of $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras to groups: The operation $\underline{\mu}$ will be referred to as multiplication and denoted simply by juxtaposition; that is, $\alpha\beta\underline{\mu}$ will be written as $\alpha\beta$. The familiar symbol $\prod$ before an ordered set of elements of a $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra will replace $\underline{\mu}'$ after the set, and α and β will be called factors of the product $\alpha\beta$. Similarly, the expressions $\alpha\beta\underline{\gamma}$, $\{\alpha_i : i \in \underline{m}\}\underline{\gamma}'$ and $\alpha(\beta\underline{\gamma})^n$ will be written $[\alpha, \beta]$, $[\alpha_0, \alpha_1, \ldots, \alpha_{m-1}]$ and $[\alpha, n\beta]$ respectively, and α and β will be called entries of the commutator $[\alpha, \beta]$. However the expression $\alpha\underline{\pi}$ is retained, because the expression α^p now means $\alpha(\alpha\underline{\mu})^{p-1}$.

The laws, in this notation, are:

(i) $$(\xi\eta)\zeta \overset{\underline{\underline{V}}}{=} \xi(\eta\zeta) ;$$

(ii) $$\eta\xi \overset{\underline{\underline{V}}}{=} \xi\eta[\eta, \xi] ;$$

(iii) $$\xi\underline{\pi} \overset{\underline{\underline{V}}}{=} \xi^p ;$$

(iv) $$\left[\prod\{\xi_d : d \in \Delta\}, \prod\{\eta_g : g \in \Gamma\}\right] \overset{\underline{\underline{V}}}{=} \prod\{[\xi_d, \eta_g] : (d, g) \in \Delta \times \Gamma\} \prod\{\zeta_t : t \in \Theta_1\} ;$$

(v) $$\left(\prod\{\xi_d : d \in \Delta\}\right)\underline{\pi}^m \overset{\underline{\underline{V}}}{=} \prod\{\zeta_t : t \in \Theta_2\} ;$$

(vi) (a) $$[\xi\underline{\pi}^m, \eta] \overset{\underline{\underline{V}}}{=} [\xi, \eta]\underline{\pi}^m \prod\{\zeta_t : t \in \Theta_3\} ,$$

(b) $$[\xi, \eta\underline{\pi}^m] \overset{\underline{\underline{V}}}{=} \prod\{\zeta_t : t \in \Theta_4\}[\xi, \eta]\underline{\pi}^m$$

where the symbols and sets are as described earlier.

The first consequence of these laws to be worked out deals with the type of substitution considered in Lemmas 1.1.5 and 1.2.2, in the special case where φ is a cpp-word, only one initial subword of φ is mapped non-identically, and that subword is mapped to a product of cpp-words.

1.3.2 LEMMA. *Let* $\varphi = \varphi(\xi_0, \ldots, \xi_{m-1}, \xi_m)$ *be a cpp-word in* B *such that* $\text{c-wt}(\varphi, \xi_m) = 1$. *Let* β *be an arbitrary cpp-word in* B , *and* $\{\alpha_l : l \in \underline{n}\}$ *a set of cpp-words such that* $\beta \leq' \alpha = \prod\{\alpha_l : l \in \underline{n}\}$. *Then every group-like variety* $\underline{\underline{V}}$ *has a law of the form*

$$\varphi(\xi_0, \ldots, \xi_{m-1}, \alpha) \overset{\underline{\underline{V}}}{=} \prod\{\varphi(\xi_0, \ldots, \xi_{m-1}, \alpha_l) : l \in \underline{n}\} \prod\{\zeta_d : d \in \Delta\}$$

where for each d *in* Δ , δ_d *is a cpp-word such that*

$$\varphi(\xi_0, \ldots, \xi_{m-1}, \beta) <' \delta_d ,$$

for h *in* $\underline{m}$,

$$\text{c-wt}(\delta_d, \xi_h) \geq \text{c-wt}(\varphi, \xi_h) + 2\ \text{c-wt}(\beta, \xi_h)\ ,$$

and

$$\text{c-wt}(\delta_d, \xi_m) \geq 2\ \text{c-wt}(\beta, \xi_m)\ .$$

Proof. Proceed by induction on the number of symbols $\underline{\gamma}$ or $\underline{\pi}$ in the word φ .

Case (i). If φ has no symbols $\underline{\gamma}$ or $\underline{\pi}$, then $\varphi = \xi_m$ and $\varphi(\xi_0, \ldots, \xi_{m-1}, \alpha) = \alpha = \prod \{\alpha_l : l \in \underline{n}\}$. The statement of the lemma clearly holds true, with $\Delta = \emptyset$.

For the remaining cases, in which the word φ terminates in either $\underline{\gamma}$ or $\underline{\pi}$, assume inductively that the result is already established for all words with fewer symbols equal to $\underline{\gamma}$ or $\underline{\pi}$ than has φ .

Case (ii). If $\varphi = \psi_1\psi_2\underline{\gamma} = [\psi_1, \psi_2]$, then either $\text{c-wt}(\psi_1, \xi_m) = 1$ and $\text{c-wt}(\psi_2, \xi_m) = 0$ or the roles of ψ_1 and ψ_2 are reversed. Assume the former; the proof is essentially the same in either case. By the inductive hypothesis,

$$\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha) \stackrel{\underline{v}}{=} \prod \{\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha_l) : l \in \underline{n}\} \prod \{\zeta_d : d \in \Delta_1\}$$

where for each d in Δ_1 ,

$$\psi_1(\xi_0, \ldots, \xi_{m-1}, \beta) <' \delta_d\ ,$$

for all h in $\underline{m}$,

$$\text{c-wt}(\delta_d, \xi_h) \geq \text{c-wt}(\psi_1, \xi_h) + 2\ \text{c-wt}(\beta, \xi_h)\ ,$$

and

$$\text{c-wt}(\delta_d, \xi_m) \geq 2\ \text{c-wt}(\beta, \xi_m)\ .$$

Now

$$\begin{aligned}
&\varphi(\xi_0, \ldots, \xi_{m-1}, \alpha) \\
&\quad = [\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha), \psi_2(\xi_0, \ldots, \xi_{m-1}, \alpha)] \\
&\quad \stackrel{\underline{v}}{=} \left[\prod \{\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha_l) : l \in \underline{n}\} \prod \{\zeta_d : d \in \Delta_1\}, \psi_2\right]\ .
\end{aligned}$$

Application of law (iv) to the last expression above gives:

$$\stackrel{\mathrm{v}}{=} \prod \{[\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha_l), \psi_2] : l \in \underline{n}\} \prod \{[\zeta_d, \psi_2] : d \in \Delta_1\}$$

$$\prod \{\zeta_d : d \in \Delta_2\}$$

where for d in Δ_2, there exist elements ϑ_1 and ϑ_2 in the set $\{\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha_l) : l \in \underline{n}\} \cup \{\delta_d : d \in \Delta_1\}$ such that

$$[\vartheta_1, \psi_2, \vartheta_2] \leq' \delta_d .$$

For each l in $\underline{n}$,

$$[\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha_l), \psi_2] = \varphi(\xi_0, \ldots, \xi_{m-1}, \alpha_l) .$$

For each d in Δ_1, Corollary 1.1.6 shows that

$$\varphi(\xi_0, \ldots, \xi_{m-1}, \beta) = [\psi_1(\xi_0, \ldots, \xi_{m-1}, \beta), \psi_2] <' [\delta_d, \psi_2]$$

and Lemma 1.1.5 that for h in $\underline{m}$,

$$\begin{aligned} \text{c-wt}([\zeta_d, \psi_2], \xi_h) &\geq \text{c-wt}(\psi_1, \xi_h) + 2\ \text{c-wt}(\beta, \xi_h) + \text{c-wt}(\psi_2, \xi_h) \\ &\geq \text{c-wt}(\varphi, \xi_h) + 2\ \text{c-wt}(\beta, \xi_h) \end{aligned}$$

and that

$$\text{c-wt}([\zeta_d, \psi_2], \xi_m) \geq 2\ \text{c-wt}(\beta, \xi_m) .$$

Thus each word of the form $[\zeta_d, \psi_2]$ for d in Δ_1 is of the required form. If, for d in Δ_2, one of the corresponding words ϑ_1 or ϑ_2 is from the set $\{\zeta_d : d \in \Delta_1\}$, then *a fortiori* ζ_d satisfies the same conditions. For all other d in Δ_2, $\delta_d = [\vartheta_1, \psi_2, \vartheta_2]$ where both ϑ_1 and ϑ_2 are in the set $\{\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha_l) : l \in \underline{n}\}$; and again it is routine to check that the appropriate conditions are satisfied.

Case (iii). If $\varphi = \psi\underline{\pi}$, then $\text{c-wt}(\psi, \xi_m) = 1$. If for some non-negative integer n, $\psi = \xi_{m}\underline{\pi}^{n}$, then the required result follows immediately from law (v). Otherwise, by the inductive hypothesis,

$$\psi(\xi_0, \ldots, \xi_{m-1}, \alpha) \stackrel{\mathrm{v}}{=} \prod \{\psi(\xi_0, \ldots, \xi_{m-1}, \alpha_l) : l \in \underline{n}\} \prod \{\zeta_d : d \in \Delta_3\}$$

where for each d in Δ_3 ,

$$\psi(\xi_0, \ldots, \xi_{m-1}, \beta) <' \zeta_d ,$$

$$\forall h \in \underline{m} \quad \text{c-wt}(\delta_d, \xi_h) \geq \text{c-wt}(\psi, \xi_h) + 2\ \text{c-wt}(\beta, \xi_h) ,$$

and

$$\text{c-wt}(\delta_d, \xi_m) \geq 2\ \text{c-wt}(\beta, \xi_m) .$$

Application of law (v) with $m = 1$ gives

$$\varphi(\xi_0, \ldots, \xi_{m-1}, \alpha) \overset{\text{v}}{=} \prod \left\{ \zeta_{d^-}{}^{\pi\,l(d)} : d \in \Delta_4 \right\}$$

where for d in Δ_4 , ζ_d is a cpp-word; the set $\left\{ \zeta_{d^-}{}^{\pi\,l(d)} : d \in \Delta_4 \right\}$ contains $\{\varphi(\xi_0, \ldots, \xi_{m-1}, \alpha_l) : l \in \underline{n}\}$ and $\{\zeta_{d^-}{}^{\pi} : d \in \Delta_3\}$ as subsets; and for those d in Δ_4 such that $\zeta_{d^-}{}^{\pi\,l(d)}$ is not in one of the sets already mentioned, either $l(d) = 1$ and ζ_d has at least two distinct subwords equal to words from the set

$$\{\psi(\xi_0, \ldots, \xi_{m-1}, \alpha_l) : l \in \underline{n}\} \cup \{\delta_d : d \in \Delta_3\} ,$$

or $l(d) = 0$ and δ_d has at least p distinct subwords equal to words from the same set. In each of these cases it is easily checked that $\zeta_{d^-}{}^{\pi\,l(d)}$ satisfies the required conditions. □

1.4 Laws connecting a word with a product of cpp-words

The laws referred to in the heading of this section are the first step toward the aim of finding laws in an arbitrary group-like variety which link an arbitrary given word with the product of a set of scpp-words with appropriately heavy weights. In subsection 1.4.1 the complexity of a word (a concept not used outside this section) is defined. The parts of Lemma 1.4.2 state the main results of this section. Lemma 1.4.3 is a similar but cruder result, which is nevertheless useful as a starting-point for inductive arguments in Section 1.5.

1.4.1 COMPLEXITY

A proper subword β of a word α in B is called a $\underline{\mu}'$*-subword* if and only if

(i) the last symbol in β is $\underline{\mu}$, and

(ii) the last symbol in the subword which immediately follows β in the maximal sequence of subwords linking β with α , is either $\underline{\gamma}$ or $\underline{\pi}$. □

The *complexity* of a word α is now defined by:

$$\mathrm{comp}(\alpha) = \sum \{k(\rho,\alpha) + l(\rho,\alpha) : \rho \text{ is a } \underline{\mu}'\text{-subword of } \alpha\} ,$$

where k and l are, as described in 1.2.1, functions on B defined similarly to those defined on A at the beginning of 1.1.3. □

A word has complexity zero if and only if it is a (possibly trivial) product of cpp-words.

1.4.2 LEMMA. *Let* $\underline{\underline{V}}$ *be a group-like variety. Then:*

(a) corresponding to an arbitrary word α *in* B *, there exists an ordered set* Δ *of cpp-words in* B *such that* $\alpha \leq' \Pi\Delta$ *and* $\alpha \stackrel{V}{=} \Pi\Delta$ *;*

(b) corresponding to a word α *in* B *whose last symbol is* $\underline{\gamma}$ *, there exists an ordered set* Δ *of cpp-words in* B *, all ending in* $\underline{\gamma}$ *, such that* $\alpha \leq' \Pi\Delta$ *and* $\alpha \stackrel{V}{=} \Pi\Delta$ *; and*

(c) corresponding to a word α *in* B *which does not contain the symbol* $\underline{\pi}$ *, there exists an ordered set* Δ *of c-words in* B *such that* $\alpha \leq' \Pi\Delta$ *and* $\alpha \stackrel{V}{=} \Pi\Delta$ *.*

Proof. Proceed by induction on $\mathrm{comp}(\alpha)$. If $\mathrm{comp}(\alpha) = 0$, then in each part of the lemma it is clear that the given word α already is in the required form. Otherwise, α has a subword β satisfying the conditions: either

(i) $\beta = \beta_1\beta_2\underline{\mu}\underline{\pi}$, or

(ii) $\beta = \beta_1\beta_2\underline{\mu}\beta_3\underline{\gamma}$ or $\beta = \beta_3\beta_1\beta_2\underline{\mu}\underline{\gamma}$;

where in either case β_1 and β_2 , and in case (ii), β_3 , have complexity zero.

(That a subword β of one of the forms (i) or (ii) exists follows from the fact that $\mathrm{comp}(\alpha) \neq 0$; that the subwords β_i of β have complexity zero can be arranged inductively, because if one of the β_i had non-zero complexity, it would itself have a subword β of strictly smaller complexity, of one of the forms (i) or (ii).)

Since β_1 and β_2 are themselves products of cpp-words, there exists, by law (i), an ordered set Θ of cpp-words such that $\beta_1\beta_2\underline{\mu} \leq' \Pi\Theta$ and $\beta_1\beta_2\underline{\mu} \overset{v}{=} \Pi\Theta$. By law (v) in case (i) and law (iv) in case (ii), it now follows that there exists an ordered set Γ of cpp-words such that $\beta \leq' \Pi\Gamma$ and $\beta \overset{v}{=} \Pi\Gamma$. In the situation of part (c) of the lemma, no word in Γ contains a symbol $\underline{\pi}$; and if the word β ends in the symbol $\underline{\gamma}$ (that is, in case (ii)) then each word in Γ also ends in the symbol $\underline{\gamma}$.

Suppose that $\alpha = \alpha(\xi_0, \ldots, \xi_{z-1})$. Since β is a subword of α , there exists a word $\alpha' = \alpha'(\xi_0, \ldots, \xi_{z-1}, \xi_z)$ such that

$$\alpha = \alpha'(\xi_0, \ldots, \xi_{z-1}, \beta) \overset{v}{=} \alpha'(\xi_0, \ldots, \xi_{z-1}, \Pi\Gamma) ;$$

and from Lemma 1.2.2, $\alpha \leq' \alpha'(\xi_0, \ldots, \xi_{z-1}, \Pi\Gamma)$. If the word α ends in the symbol $\underline{\gamma}$, then either α' also ends in $\underline{\gamma}$ or $\alpha' = \xi_z$ and $\alpha = \beta \overset{v}{=} \Pi\Gamma$. In the latter case, the proof of part (b) is complete. In the former case for part (b), and in all cases for parts (a) and (c), the expression $\alpha'(\xi_0, \ldots, \xi_{z-1}, \Pi\Gamma)$ satisfies all the conditions required of α in the statement of the lemma and has strictly lower complexity. From the inductive hypothesis then, there exists a set Δ of cpp-words satisfying the conditions of the appropriate part of the lemma such that

$$\alpha \leq' \alpha'(\xi_0, \ldots, \xi_{z-1}, \Pi\Gamma) \leq' \Pi\Delta$$

and

$$\alpha \overset{\underline{V}}{=} \alpha\,(\xi_0, \ldots, \xi_{z-1}, \Pi\Gamma) \overset{\underline{V}}{=} \Pi\Delta \; .$$

As both relations $\leq'$ and $\overset{\underline{V}}{=}$ are transitive, this completes the proof. □

1.4.3 LEMMA. *Corresponding to a group-like variety* $\underline{V}$ *and an arbitrary word* $\alpha = \alpha(\xi_0, \ldots, \xi_{z-1})$ *in* B *, there exists an ordered set* Δ *of c-words in* B *such that for all* δ *in* Δ *and all* i *in* $\underline{z}$ *,*

$$\text{c-wt}(\delta, \xi_i) \geq \text{c-wt}(\alpha, \xi_i)$$

and $\alpha \overset{\underline{V}}{=} \Pi\Delta$.

Proof. Proceed by induction on the number of symbols $\underline{\pi}$ in the word α . If there are none, then the result follows immediately from Lemma 1.4.2 (c). Otherwise, α must have a subword $\beta\underline{\pi}$ where β does not contain the symbol $\underline{\pi}$. Law (iii) shows that

$$\beta\underline{\pi} \overset{\underline{V}}{=} \beta(\beta\underline{\mu})^{p-1} = \beta^p \; ;$$

and although the weight functions $w^e_{a,b}$ with either $e = p$ or $b \geq 1$ take greater values at $\beta\underline{\pi}$ than at β^p, it remains true that, for all ξ in Ξ ,

$$\text{c-wt}(\beta\underline{\pi}, \xi) = \text{c-wt}(\beta^p, \xi) \; .$$

Let $\alpha' = \alpha'(\xi_0, \ldots, \xi_{z-1}, \xi_z)$ be the word in B with $\text{c-wt}(\alpha', \xi_z) = 1$ such that

$$\alpha = \alpha'(\xi_0, \ldots, \xi_{z-1}, \beta\underline{\pi}) \overset{\underline{V}}{=} \alpha'(\xi_0, \ldots, \xi_{z-1}, \beta^p) \; .$$

The last word above has fewer symbols $\underline{\pi}$ then has α , but has the same c-weight in every element of Ξ . Hence the required result follows by induction.

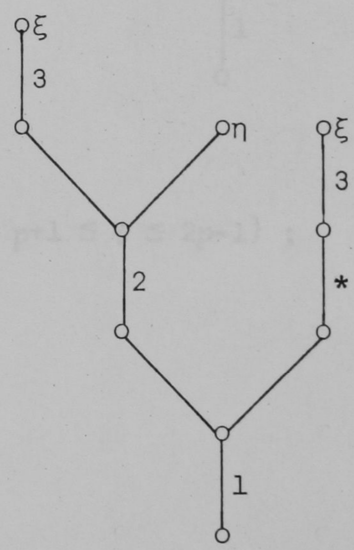
1.5 Laws connecting a word with a product of scpp-words

In sub-section 1.5.1, subword arrays of a cpp-word are introduced, and in 1.5.2, 1.5.3 and 1.5.4 some of their simpler properties are established. Subword arrays provide a useful, though complicated, tool for the proof of the main result of this section, Lemma 1.5.5, which deals with a law in a group-like variety linking an arbitrary cpp-word with a product of simple cpp-words. In its application to groups, this theorem expresses an arbitrary cpp-element as a product of p-power powers of commutators, and the information it gives linking the weights and the powers in such an expression appears to be best possible in general, though a proof of this is not attempted. The process is illustrated in Example 1.5.6. However, this detailed information is inconveniently complicated, and for most purposes the simplified statement of Theorem 1.5.7 is more useful.

1.5.1 SUBWORD ARRAYS

A subword array of a cpp-word φ is a family (constructed as described below) of subwords of φ whose symbols $\underline{\pi}$ are labelled in a suitable way.

A $\underline{\pi}$-*labelled cpp-word* is a cpp-word, φ say, to which has been attached, at each symbol $\underline{\pi}$, a symbol from the set $\{*\} \cup Z^{+}$ in accordance with the rule that for some fixed non-negative integer n, the final symbol $\underline{\pi}$ in each subword of the form $\rho\underline{\pi}$ has attached to it either the integer $l(\rho, \varphi) + n$ or the symbol $*$. A *basic* $\underline{\pi}$-labelling of φ is one in which $n = 0$.



For example, $\xi\overset{3}{\underline{\pi}}\eta\gamma\overset{2}{\underline{\pi}}\xi\overset{3}{\underline{\pi}}\overset{*}{\underline{\pi}}\gamma\overset{1}{\underline{\pi}}$ and $\xi\overset{6}{\underline{\pi}}\eta\gamma\overset{*}{\underline{\pi}}\xi\overset{*}{\underline{\pi}}\overset{*}{\underline{\pi}}\gamma\overset{*}{\underline{\pi}}$ are $\underline{\pi}$-labelled cpp-words (the former being basic), but

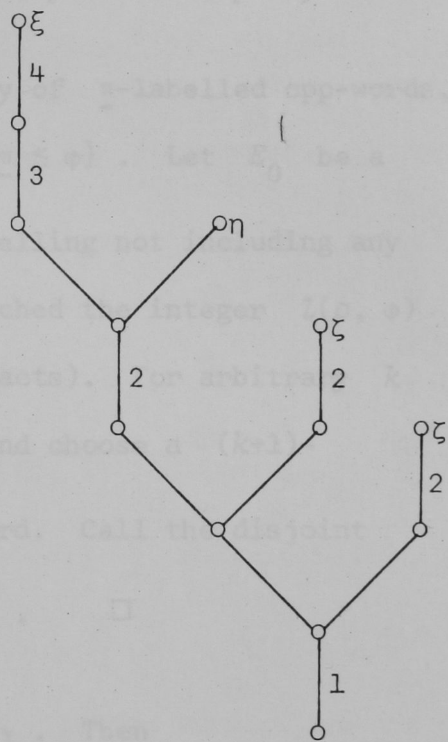
$\xi\overset{2}{\underline{\pi}}\eta\gamma\overset{*}{\underline{\pi}}\xi\overset{*}{\underline{\pi}}\overset{*}{\underline{\pi}}\gamma\overset{*}{\underline{\pi}}$ and $\xi\overset{4}{\underline{\pi}}\eta\gamma\overset{*}{\underline{\pi}}\xi\overset{*}{\underline{\pi}}\overset{2}{\underline{\pi}}\gamma\overset{*}{\underline{\pi}}$ are not. □

Let φ be a $\underline{\pi}$-labelled cpp-word, and k a positive integer. Partition the set of symbols $\underline{\pi}$ in φ labelled with the integer k into two (possibly empty) subsets, S_1 and S_2 . A *k-propogation* of φ corresponding to this partition is a family (that is, repetitions are allowed) of $\underline{\pi}$-labelled words, consisting of:

(i) the labelled word obtained from φ by replacing, for all $\underline{\pi}$ in S_1 , the symbol k attached to $\underline{\pi}$ by a symbol $*$; and

(ii) $p - 1$ distinctly-identified copies of each labelled subword ρ of φ on which a symbol $\underline{\pi}$ in S_2 acts. □

For example, let $\varphi = \xi\underline{\overset{4}{\pi}\overset{3}{\pi}}\eta\underline{\gamma\overset{2}{\pi}}\zeta\underline{\overset{2}{\pi}\gamma}\zeta\underline{\overset{2}{\pi}\gamma\overset{1}{\pi}}$, and $k = 2$. Consider first the partition such such that S_1 contains all symbols $\overset{2}{\underline{\pi}}$ in φ , and S_2 is empty.

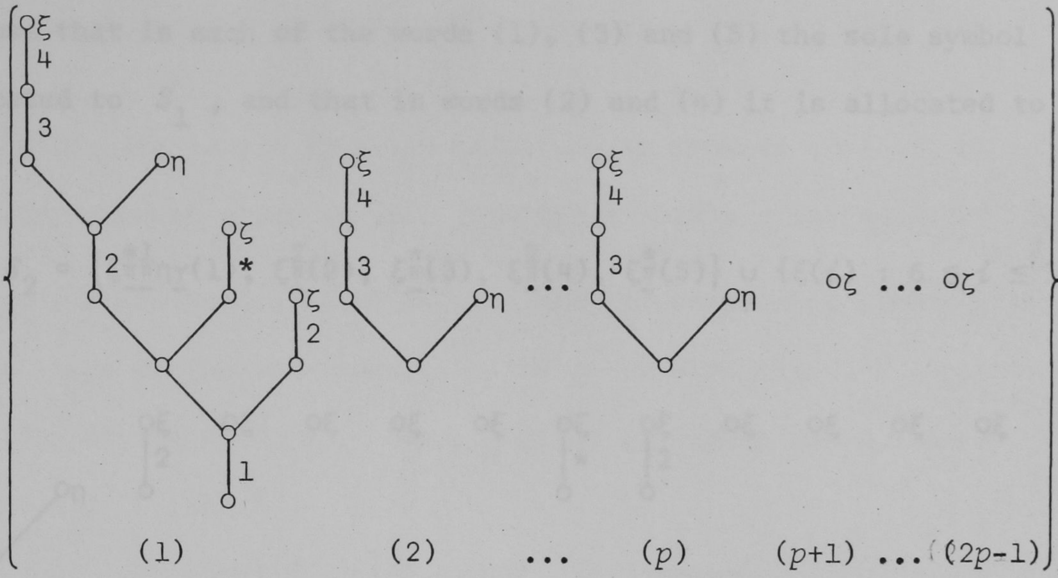


A corresponding 2-propogation of φ is simply:

$$\{\xi\underline{\overset{4}{\pi}\overset{3}{\pi}}\eta\underline{\gamma\overset{*}{\pi}}\zeta\underline{\overset{*}{\pi}\gamma}\zeta\underline{\overset{*}{\pi}\gamma\overset{1}{\pi}}\} .$$

Consider another partition of φ in which S_1 contains the second symbol $\overset{2}{\underline{\pi}}$, and S_2 contains the first and the third. A corresponding 2-propogation of φ is

$$\{\xi\underline{\overset{4}{\pi}\overset{3}{\pi}}\eta\underline{\gamma\overset{2}{\pi}}\zeta\underline{\overset{*}{\pi}\gamma}\zeta\underline{\overset{2}{\pi}\gamma\overset{1}{\pi}}(1)\} \cup \{\xi\underline{\overset{4}{\pi}\overset{3}{\pi}}\eta\underline{\gamma}(i) : 2 \le i \le p\} \cup \{\zeta(i) : p{+}1 \le i \le 2p{-}1\} ;$$



. □

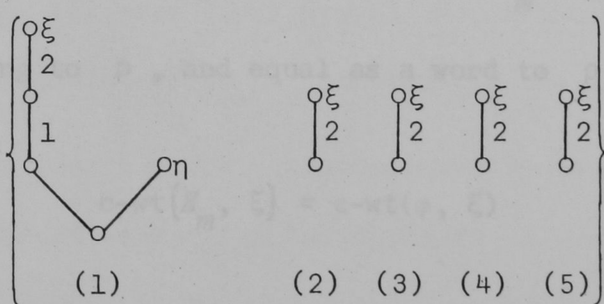
A *subword array* of a cpp-word φ is a family of $\underline{\pi}$-labelled cpp-words, constructed as follows. Let $h = \max\{l(\rho, \varphi) : \rho\underline{\pi} \le \varphi\}$. Let E_0 be a family whose only member is φ with a basic labelling not including any stars (that is, to each symbol $\underline{\pi}$ in φ is attached the integer $l(\rho, \varphi)$ where ρ is the subword on which the symbol $\underline{\pi}$ acts). For arbitrary k in $\underline{h}$, suppose that E_k has been constructed, and choose a $(k+1)$-propogation of each copy in E_k of a labelled word. Call the disjoint union of these families E_{k+1} . Finally, $E = E_h$. □

As an example, suppose $p = 5$ and $\varphi = \xi\underline{\pi}\underline{\pi}\eta\underline{\gamma}$. Then

$$E_0 = \{\xi\overset{2}{\underline{\pi}}\overset{1}{\underline{\pi}}\eta\underline{\gamma}(1)\} .$$

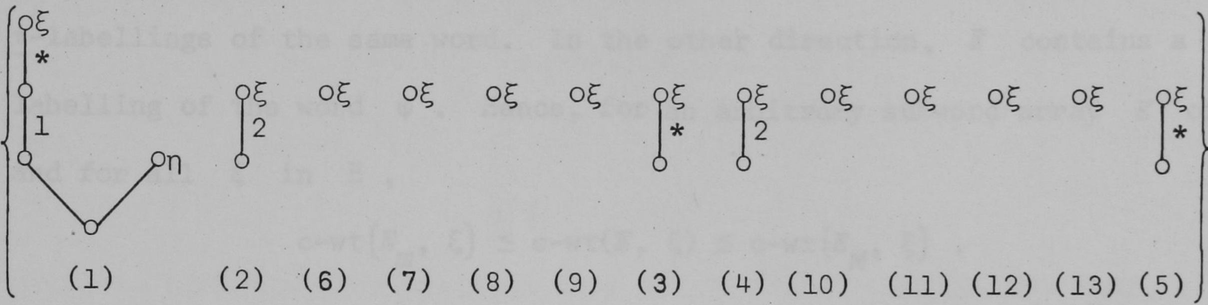
Now, for $k = 1$, choose $S_1 = \emptyset$ and S_2 to contain the only symbol $\overset{1}{\underline{\pi}}$ in the copy of φ in E_0 . Then

$$E_1 = \{\xi\overset{2}{\underline{\pi}}\overset{1}{\underline{\pi}}\eta\underline{\gamma}(1)\} \cup \{\xi\overset{2}{\underline{\pi}}(i) : 2 \le i \le p\} ,$$



Suppose that in each of the words (1), (3) and (5) the sole symbol $\overset{2}{\underline{\pi}}$ is allocated to S_1, and that in words (2) and (4) it is allocated to S_2. Then

$$E_2 = \{\xi\overset{*}{\underline{\pi}}\overset{1}{\underline{\pi}}\eta\underline{\gamma}(1),\ \xi\overset{2}{\underline{\pi}}(2),\ \xi\overset{*}{\underline{\pi}}(3),\ \xi\overset{2}{\underline{\pi}}(4),\ \xi\overset{*}{\underline{\pi}}(5)\} \cup \{\xi(i) : 6 \le i \le 13\},$$



□

Suppose $E = \{\rho(i) : i \in \underline{m}\}$ is a subword array for a cpp-word. For each ξ in Ξ, define

$$\text{c-wt}(E, \xi) = \sum \{\text{c-wt}(\rho(i), \xi) : i \in \underline{m}\}$$

and

$$\text{c-wt}(E) = \sum \{\text{c-wt}(\rho(i)) : i \in \underline{m}\} = \sum \{\text{c-wt}(E, \xi) : \xi \in \Xi\}.$$

Define $n(E)$ to be the total number of starred symbols $\underline{\pi}$ in labelled words $\rho(i)$ for i in $\underline{m}$. □

Corresponding to a given cpp-word φ, let E_m be the subword array for φ in which every symbol $\underline{\pi}$ which occurs is starred; thus E_m contains only one $\underline{\pi}$-labelled word, a $\underline{\pi}$-labelling of φ itself. Let E_M be the subword array for φ such that no symbol $\underline{\pi}$ in a word in E_M is starred. Corresponding to each initial subword ρ of φ there are precisely $p^{l(\rho,\varphi)}$ distinct copies of words in E_M each containing a subword corresponding to ρ, and equal as a word to ρ. Hence, for all ξ in Ξ,

$$\text{c-wt}(E_m, \xi) = \text{c-wt}(\varphi, \xi)$$

and

$$\text{c-wt}(E_M, \xi) = \text{cpp-wt}(\varphi, \xi)\ . \qquad \square$$

It is easy to see from the construction process that if E is an arbitrary subword array of φ, then there exists a subfamily of E_M in one-one correspondence with E such that corresponding elements are $\underline{\pi}$-labellings of the same word. In the other direction, E contains a labelling of the word φ. Hence, for an arbitrary subword array E of φ and for all ξ in Ξ,

$$\text{c-wt}(E_m, \xi) \le \text{c-wt}(E, \xi) \le \text{c-wt}(E_M, \xi)\ .$$

1.5.2 LEMMA. *Given a subword array E for a cpp-word φ, and an integer l such that $0 \le l \le n(E)$, there exists a subword array E' for φ such that*

$$n(E') = n(E) - l\ ,$$

$$\text{c-wt}(E) + l(p-1) \le \text{c-wt}(E') \le p^l\ \text{c-wt}(E)\ ,$$

and

$$\forall\ \xi \in \Xi\ , \quad \text{c-wt}(E, \xi) \le \text{c-wt}(E', \xi) \le p^l\ \text{c-wt}(E, \xi)\ .$$

Proof. The case $l = 0$ is trivial. The form of the statement is such that it follows immediately by induction from its special case $l = 1$.

Suppose $l = 1$; then $n(E) \ge 1$ and there exists at least one starred symbol $\underline{\pi}$ in a copy in E of a labelled word. Consequently there is a copy in E of a labelled word with a subword $\rho\overset{*}{\underline{\pi}}$ such that neither ρ nor a copy of a subword derived from ρ contains a starred symbol $\underline{\pi}$. For the corresponding "plain" cpp-word ρ, let F_M be that subword array in which no symbol $\underline{\pi}$ is starred.

Now a subword array E' for φ is constructed in the same way as was the array E, except that the final symbol $\underline{\pi}$ of the copy of $\rho\underline{\pi}$ under consideration is allocated to the S_2 instead of the S_1 subset at the appropriate stage of construction, and consequently is not starred in E'. Further, $p - 1$ subword arrays for the cpp-word ρ, all chosen to equal

F_M , are constructed in E' in addition to copies in one-one correspondence with copies of labellings of the same word in E . Thus, for all ξ in Ξ ,

$$\text{c-wt}(E', \xi) = \text{c-wt}(E, \xi) + (p-1)\text{c-wt}\left(F_M, \xi\right) ;$$

by adding these,

$$\text{c-wt}(E') = \text{c-wt}(E) + (p-1)\text{c-wt}\left(F_M\right) ;$$

and

$$n(E') = n(E) - 1 .$$

From the choice of the original copy of ρ in E , it follows that there is a one-one correspondence between elements of F_M and copies of labellings of equal (except possibly in the case of ρ) words in a subset of E . Hence for all ξ in Ξ ,

$$0 \le \text{c-wt}\left(F_M, \xi\right) \le \text{c-wt}(E, \xi)$$

and for at least one ξ in Ξ ,

$$1 \le \text{c-wt}\left(F_M, \xi\right) \le \text{c-wt}(E, \xi) ;$$

adding these as before gives

$$1 \le \text{c-wt}\left(F_M\right) \le \text{c-wt}(E) .$$

When these relations are substituted into the earlier equalities, the required results, namely:

$$\forall\ \xi \in \Xi ,\ \ \text{c-wt}(E, \xi) \le \text{c-wt}(E', \xi) \le p\ \text{c-wt}(E, \xi)$$

and

$$\text{c-wt}(E) + (p-1) \le \text{c-wt}(E') \le p\ \text{c-wt}(E) ,$$

are obtained. □

1.5.3 LEMMA. *Let* φ *be a cpp-word and* E_m *a subword array for* φ *in which every symbol* $\underline{\pi}$ *occurring is starred. If* E *is an arbitrary subword array for* φ *, then*

$$\text{c-wt}(E) \ge \text{c-wt}\left(E_m\right) + (p-1)\left(n\left(E_m\right) - n(E)\right) .$$

Proof. If $n(E_m) \le n(E)$, then the conclusion is obvious from the remarks at the end of 1.5.1. Otherwise, in the construction of E , at least $n(E_m) - n(E)$ symbols $\underline{\pi}$ in the basically-labelled copy of φ are left unstarred; and correspondingly at least $(p-1)\left(n(E_m)-n(E)\right)$ copies of words, each of c-weight at least one, are included in E in addition to the basically-labelled copy of φ . □

A final preliminary result, Lemma 1.5.4, is of interest in interpreting, rather than in proving, Lemma 1.5.5.

1.5.4 LEMMA. *Let* E *be a subword array for a word* φ *, and let* κ *be a c-word and* h *a non-negative integer such that*

$$\forall\ \xi \in \Xi\ ,\quad \text{c-wt}(\kappa,\ \xi) \ge \text{c-wt}(E,\ \xi)$$

and

$$h \ge n(E)\ .$$

Then $\varphi \le' \kappa\underline{\pi}^h$.

Proof. The required inequality dealing with c-weight in an arbitrary element of Ξ is immediate:

$$\forall\ \xi \in \Xi\ ,\quad \text{c-wt}\left(\kappa\underline{\pi}^h,\ \xi\right) \ge \text{c-wt}(E,\ \xi) \ge \text{c-wt}(\varphi,\ \xi)\ .$$

Since $h \ge n(E)$, it follows from Lemma 1.5.2 that there exists a subword array, E' say, for φ such that $n(E') = 0$ and

$$\forall\ \xi \in \Xi\ ,\quad \text{c-wt}(E,\ \xi) \le \text{c-wt}(E',\ \xi) \le p^h\ \text{c-wt}(E,\ \xi)\ .$$

Near the end of 1.5.1 it is pointed out that such an array has the property that for all ξ in Ξ ,

$$\text{c-wt}(E',\ \xi) = \text{cpp-wt}(\varphi,\ \xi)\ .$$

Hence, for all ξ in Ξ ,

$$\begin{aligned}\text{cpp-wt}\left(\kappa\underline{\pi}^h,\ \xi\right) &= p^h\ \text{c-wt}(\kappa,\ \xi)\\ &\geq p^h\ \text{c-wt}(E,\ \xi)\\ &\geq \text{c-wt}(E',\ \xi)\\ &= \text{cpp-wt}(\varphi,\ \xi)\ ,\end{aligned}$$

as required.

From this it follows automatically that for all integers a and b such that $a \geq b \geq 0$,

$$w^p_{a,b}\left(\kappa\underline{\pi}^h\right) \geq w^p_{a,b}(\varphi)\ .$$

Finally, note that

$$\begin{aligned}w^1_{a,b}\left(\kappa\underline{\pi}^h\right) &= a\ \text{c-wt}(\kappa) + bh\\ &\geq a\ \text{c-wt}(E) + bn(E)\ .\end{aligned}$$

Let E_m be the subword array for φ in which every symbol $\underline{\pi}$ is starred, so that

$$w^1_{a,b}(\varphi) = a\ \text{c-wt}\left(E_m\right) + bn\left(E_m\right)\ .$$

If $n(E) \geq n\left(E_m\right)$, then the required inequality

$$w^1_{a,b}\left(\kappa\underline{\pi}^h\right) \geq w^1_{a,b}(\varphi)$$

is immediate. If $n(E) < n\left(E_m\right)$, then Lemma 1.5.3 gives

$$\text{c-wt}(E) \geq \text{c-wt}\left(E_m\right) + (p-1)\left(n\left(E_m\right)-n(E)\right)\ ,$$

whence

$$\begin{aligned}w^1_{a,b}\left(\kappa\underline{\pi}^h\right) &\geq a\left(\text{c-wt}\left(E_m\right)+(p-1)\left(n\left(E_m\right)-n(E)\right)\right) + bn(E)\\ &= a\ \text{c-wt}\left(E_m\right) + bn\left(E_m\right) + \left(a(p-1)-b\right)\left(n\left(E_m\right)-n(E)\right)\\ &\geq w^1_{a,b}\ ,\end{aligned}$$

as required. □

1.5.5 LEMMA. *Given a group-like variety* $\underline{V}$ *and a cpp-word* φ *, there* *exist ordered sets* $\{\kappa_g : g \in \Gamma\}$ *of c-words,* $\{h(g) : g \in \Gamma\}$ *of non-negative integers, and* $\{E_g : g \in \Gamma\}$ *of subword arrays of* φ *, such that*

$$\varphi \overset{\mathrm{v}}{=} \prod \left\{\kappa_{g\underline{\pi}}{}^{h(g)} : g \in \Gamma\right\},$$

satisfying the conditions that for all g *in* Γ *and all* ξ *in* Ξ ,

$$\text{c-wt}\left(\kappa_g, \xi\right) \geq \text{c-wt}\left(E_g, \xi\right)$$

and

$$h(g) \geq n\left(E_g\right) .$$

Proof. Consider two propositions:

(a) Let φ and ψ be cpp-words, h a non-negative integer, and E a subword array for φ such that for all ξ in Ξ,

$$\text{c-wt}(\psi, \xi) \geq \text{c-wt}(E, \xi)$$

and

$$h \geq n(E) .$$

Then there exist ordered sets $\{\kappa_g : g \in \Gamma_1\}$ of c-words, $\{h(g) : g \in \Gamma_1\}$ of non-negative integers, and $\{E_g : g \in \Gamma_1\}$ of subword arrays of φ , such that

$$\psi_{\underline{\pi}}{}^{h} \overset{\mathrm{v}}{=} \prod \left\{\kappa_{g\underline{\pi}}{}^{h(g)} : g \in \Gamma_1\right\},$$

and for all g in Γ_1 and all ξ in Ξ,

(*) $$\text{c-wt}\left(\kappa_g, \xi\right) \geq \text{c-wt}\left(E_g, \xi\right)$$

and

(**) $$h(g) \geq n\left(E_g\right) .$$

(b) Let ψ be a cpp-word ending in the symbol $\underline{\gamma}$, and h a non-negative integer. Suppose that for each integer k , $0 \leq k \leq h$, to every subword array F of $\psi_{\underline{\pi}}{}^{k}$ there corresponds a subword array E of φ satisfying the conditions

$$\forall\ \xi \in \Xi\ ,\ \text{c-wt}(F, \xi) \geq \text{c-wt}(E, \xi)$$

and

$$n(F) + h - k \geq n(E) .$$

Then there exist sets $\{\kappa_g : g \in \Gamma\}$ of c-words, $\{h(g) : g \in \Gamma\}$ of non-negative integers, and $\{E_g : g \in \Gamma\}$ of subword arrays of φ, such that

$$\psi\underline{\pi}^h \overset{\mathrm{v}}{=} \prod \left\{\kappa_g\underline{\pi}^{h(g)} : g \in \Gamma\right\}$$

and for g in Γ and ξ in Ξ, conditions (*) and (**) are satisfied.

It is not hard to see that the lemma follows from proposition (b), since the given cpp-word φ may be expressed in the form $\varphi'\underline{\pi}^{h'}$ where either $\varphi' \in \Xi$ or φ' ends in the symbol $\underline{\gamma}$. In the former case, φ is already in the form required by the conclusion of the lemma; in the latter, the assumptions of proposition (b) are satisfied with $\psi = \varphi'$ and $h = h'$.

Proposition (a) is used at some points in the proof of proposition (b), and is proved first.

Proof of Proposition (a). By Lemma 1.4.3,

$$\psi \overset{\mathrm{v}}{=} \prod \{\lambda_d : d \in \Delta_1\}$$

where each λ_d is a c-word such that

$$\forall\ \xi \in \Xi ,\quad \text{c-wt}(\lambda_d, \xi) \geq \text{c-wt}(\psi, \xi) \geq \text{c-wt}(E, \xi) .$$

Hence, by law (v),

$$\psi\underline{\pi}^h \overset{\mathrm{v}}{=} \prod \left\{\kappa_g\underline{\pi}^{h(g)} : g \in \Gamma_1\right\}$$

where for g in Γ_1, κ_g is a c-word with at least $p^{h-h(g)}$ distinct subwords (not necessarily unequal as words) from the set $\{\lambda_d : d \in \Delta_1\}$, whence

$$\forall\ \xi \in \Xi ,\quad \text{c-wt}(\kappa_g, \xi) \geq p^{h-h(g)}\,\text{c-wt}(E, \xi) .$$

Now Lemma 1.5.2 is applied to show the existence of suitable subword arrays E_g. For g in Γ_1 such that $h - h(g) \geq n(E)$, Lemma 1.5.2 is used with $l = n(E)$ to show the existence of a subword array E_g such that

(*) $\quad \forall\, \xi \in \Xi$, $\text{c-wt}(\kappa_g, \xi) \geq p^{h-h(g)}\text{c-wt}(E, \xi) \geq \text{c-wt}(E_g, \xi)$

and

(**) $$h(g) \geq 0 = n(E_g) \ .$$

For g in Γ_1 such that $h - h(g) \leq n(E)$, Lemma 1.5.2 is used with $l = h - h(g)$ to construct E_g such that

(*) $\quad \forall\, \xi \in \Xi$, $\text{c-wt}(\kappa_g, \xi) \geq p^{h-h(g)}\text{c-wt}(E, \xi) \geq \text{c-wt}(E_g, \xi)$

and

(**) $$h(g) = h - (h-h(g)) = n(E_g) \ .$$

This completes the proof of proposition (a).

Proof of (b). The proof of proposition (b) is unfortunately rather more complicated, proceeding by a sequence of four nested induction arguments. It may be some consolation that the first three are easy. As a preliminary, let F_M and F_m be the subword arrays for ψ in which respectively none and every one of the symbols $\underline{\pi}$ occurring is starred; and let E'_M and E'_m be the subword arrays of φ corresponding respectively to F_M and F_m under the hypotheses of proposition (b). Let E_M and E_m be subword arrays for φ in which respectively none and every one of the symbols $\underline{\pi}$ occurring is starred. Note that E_m, E_M, E'_m and E'_M are all subword arrays of φ ; the first two depend on φ only, and the last two depend also on the word ψ .

The first induction is in the reverse direction on h . If $h \geq n(E_m)$, then the assumptions of proposition (a) are satisfied by the choice $E = E_m$, and the conclusion of proposition (a) gives the required result. From now on, only words of the form $\psi\underline{\pi}^h$ with $h < n(E_m)$ need be considered. The first inductive hypothesis will be that the result is established for all words $\psi'\underline{\pi}^{h'}$ satisfying the assumptions of the proposition and the condition that $h' > h$.

Suppose $\psi = \psi(\xi_0, \ldots, \xi_{z-1})$. The second and third inductive arguments between them cover only a finite number (at most c-wt E_M) of steps. Note that if, for all i in $\underline{z}$,

$$\text{c-wt}(\psi, \xi_i) \geq \text{c-wt}(E_M, \xi_i)$$

then *a fortiori*, for all i in $\underline{z}$,

$$\text{c-wt}(\psi, \xi_i) \geq \text{c-wt}(E'_M, \xi_i) .$$

If the latter condition is satisfied, then the assumptions of proposition (a) are satisfied by the choice $E = E'_M$, and again the required result follows. Thus only words $\psi\underline{\pi}^h$ such that for some i in $\underline{z}$, $\text{c-wt}(\psi, \xi_i) < \text{c-wt}(E'_M, \xi_j)$ need henceforth be considered. Let j be the least integer in $\underline{z}$ such that $\text{c-wt}(\psi, \xi_j) < \text{c-wt}(E'_M, \xi_j)$. Assume that the result is established for all words $\psi'\underline{\pi}^{h'}$ such that either (second inductive hypothesis, on j):

$$\text{c-wt}(\psi', \xi_i) \geq \text{c-wt}(E_M, \xi_i) \quad \text{for} \quad 0 \leq i \leq j$$

(note the final equality) or (third inductive hypothesis, in the reverse direction on $\text{c-wt}(\psi, \xi_j)$)

$$\text{c-wt}(\psi, \xi_j) < \text{c-wt}(\psi', \xi_j) .$$

For every word $\psi\underline{\pi}^h$ satisfying the conditions remaining to be considered,

$$\text{c-wt}(\psi, \xi_j) < \text{c-wt}(E'_M, \xi_j) \leq \text{c-wt}(F_M, \xi_j) = \text{cpp-wt}(\psi, \xi_j) .$$

This strict inequality shows that there exists a subword of ψ ending in the symbol $\underline{\pi}$ and having c-weight in ξ_j at least 1 . Among such subwords, let $\rho\underline{\pi}$ be one such that the value of $k(\rho, \psi)$ is minimal. The fourth induction is on this value $k(\rho, \psi)$. Since ψ itself ends in the symbol $\underline{\gamma}$, it follows that ρ is a proper subword of ψ and $k(\rho, \psi)$ is at least 1 .

In the initial case $k(\rho, \psi) = 1$, the word ψ has subwords θ and ρ such that $\text{c-wt}\left(\rho, \xi_j\right) \geq 1$ and either $\psi = [\theta, \rho\underline{\pi}]$ or $\psi = [\rho\underline{\pi}, \theta]$. The argument is essentially the same in either case; for convenience, assume the former. Law (vi) (b) with $m = 1$ gives

$$\psi \overset{\mathrm{v}}{=} \prod \left\{\delta_{g\underline{\pi}}{}^{l(g)} : g \in \Delta_2\right\}$$

where for g in Δ_2 , δ_g is a cpp-word ending in the symbol $\underline{\gamma}$, and $l(g)$ is either 0 or 1 . For those g in Δ_2 such that $l(g) = 1$, the word δ_g has at least one subword equal to each of θ and ρ ; and for those g in Δ_2 such that $l(g) = 0$, the word δ_g has at least one subword equal to θ but this time at least p distinct subwords equal to ρ . Law (v) now gives

$$\psi_{\underline{\pi}}{}^{h} \overset{\mathrm{v}}{=} \prod \left\{\delta_{d\underline{\pi}}{}^{h(d)} : d \in \Delta_3\right\}$$

where the set $\left\{\delta_{d\underline{\pi}}{}^{h(d)} : d \in \Delta_3\right\}$ has a subset equal to $\left\{\delta_{g\underline{\pi}}{}^{l(g)+h} : g \in \Delta_2\right\}$, and each other element $\delta_{d\underline{\pi}}{}^{h(d)}$ satisfies $0 \leq h(d) \leq h$, and has at least $p^{h-h(d)}$ distinct subwords equal to words from the set $\left\{\delta_{g\underline{\pi}}{}^{l(g)} : g \in \Delta_2\right\}$. The proof of the initial case $k(\rho, \psi) = 1$ of the fourth inductive hypothesis now requires a rather onerous verification that each element $\delta_{d\underline{\pi}}{}^{h(d)}$ for d in Δ_3 satisfies the assumptions of proposition (b) and also the conditions under which, by an earlier inductive hypothesis, the result is established.

Firstly, consider an element of the form $\delta_{g\underline{\pi}}{}^{l(g)+h}$ for some g in Δ_2 such that $l(g) = 1$. For some integer k such that $0 \leq k \leq 1+h$, let G be a subword array for $\delta_g\underline{\pi}^k$; an appropriate corresponding subword array E for φ is required. As an intermediate step, construct a subword array F , for ψ if $k = 0$, and for $\psi\underline{\pi}^{k-1}$ if $1 \le k \le 1+h$.

If $k = 0$, then choose one fixed subword of δ_g equal as a word to each of θ and ρ . In constructing F_1 , use a 1-propogation of the basic labelling of $\psi = [\theta, \rho\underline{\pi}]$ which allocates the final symbol $\overset{1}{\underline{\pi}}$ in the subword $\rho\overset{1}{\underline{\pi}}$ to S_1 , and each symbol $\overset{1}{\underline{\pi}}$ in θ to S_1 or S_2 according as the corresponding symbol $\underline{\pi}$ (not necessarily labelled 1) in the selected subword θ of δ_g was allocated to S_1 or S_2 in the construction of G . At each later stage of the construction of F , to each symbol $\underline{\pi}$ contained in ρ or a copy of a subword of ρ , or θ or a copy of a subword of θ , there corresponds precisely one symbol $\underline{\pi}$ contained in the corresponding selected subword ρ or θ of δ_g , or a copy of one of its subwords as appropriate. The symbol $\underline{\pi}$ is allocated to S_1 or S_2 in the construction of F according as its counterpart was allocated to S_1 or S_2 in the construction of G , and so the correspondence between symbols is maintained for later stages of the construction. Clearly

$$n(F) \le n(G) + 1 ,$$

and

$$\forall\ \xi \in \Xi ,\ \text{c-wt}(F, \xi) \le \text{c-wt}(G, \xi) .$$

If $1 \le k \le 1+h$ and G is a subword array for $\delta_g\underline{\pi}^k$, then a subword array F for $\psi\underline{\pi}^{k-1}$ is constructed. In the construction of F_n for $1 \le n \le k-1$, an exact correspondence with the construction of G_n is maintained. Thus there is a one-one correspondence between F_{k+1} and G_{k+1} ; each element of the former is a copy of a labelled word with a unique subword equal to $\psi = [\theta, \rho\underline{\pi}]$ and each element of the latter is a copy of a labelled word with a unique subword equal to $\delta_{g^-}\pi$. The final $\underline{\pi}$ in a subword $\rho\underline{\pi}$ is allocated to S_1 or S_2 in the construction of F_k according as the final $\underline{\pi}$ in the corresponding subword $\delta_{g^-}\pi$ was allocated to S_1 or S_2 in the construction of G_k . For each such symbol allocated to S_2 , $p - 1$ "new" copies of ρ are introduced into F_k , and each corresponds to a chosen subword ρ in a corresponding "new" copy of δ_g introduced into G_k . For succeeding stages of the construction, correspondence is maintained in the way already described. Thus

$$n(F) \leq n(G)$$

and

$$\forall\ \xi \in \Xi\ ,\ \ \text{c-wt}(F,\ \xi) \leq \text{c-wt}(G,\ \xi)\ .$$

Following the construction of F as in one of the two preceding paragraphs, let E be the subword array of φ corresponding to F under the hypotheses of proposition (b). Then, in either case,

$$n(G) + h + 1 - k \geq n(F) + h - k \geq n(E)\ ,$$

and

$$\forall\ \xi \in \Xi\ ,\ \ \text{c-wt}(G,\ \xi) \geq \text{c-wt}(F,\ \xi) \geq \text{c-wt}(E,\ \xi)\ .$$

Hence each term $\delta_{g^-}\pi^{l(g)+h}$ for g in Δ_2 such that $l(g) = 1$ satisfies the assumptions of proposition (b); and since it also satisfies the conditions of the first inductive hypothesis, it may be expressed in the required form.

Secondly, consider an element of the form $\delta_{g^-}\pi^{l(g)+h}$ for g in Δ_2 such that $l(g) = 0$. Let k be an integer satisfying $0 \leq k \leq h$, and let G be a subword array for $\delta_{g^-}\pi^k$. A subword array F for $\psi\underline{\pi}^k$ is constructed as follows. In the construction of F_n for $1 \leq n \leq k$, an exact correspondence with the construction of G_n is maintained; thus there is a one-one correspondence between copies in F_k of words each with a unique subword equal to $\psi = [\theta, \rho\underline{\pi}]$ and copies in G_k of words each with a unique subword equal to δ_g. In each element of F_k, the final symbol $\underline{\pi}$ of its copy of $\rho\underline{\pi}$ is allocated to S_2, so that in F_{k+1} there are $p - 1$ "new" copies of ρ. These and the subword ρ in the "old" copy are made to correspond with a chosen set of p distinct subwords (also equal to ρ) of the corresponding copy of δ_g in G_k ; and the subword θ of ψ in F_k corresponds with a chosen subword θ of the same corresponding copy of δ_g in G_k. Further construction of F is continued in accordance with this correspondence.

As before, E is chosen to be the subword array for φ corresponding, under the hypotheses of proposition (b), with the array F for $\psi\underline{\pi}^k$. Then

$$n(G) + h - k \geq n(F) + h - k \geq n(E) ,$$

and

$$\forall\ \xi \in \Xi\ ,\quad \text{c-wt}(G, \xi) \geq \text{c-wt}(F, \xi) \geq \text{c-wt}(E, \xi)\ .$$

Thus each term $\delta_g\underline{\pi}^h$ for g in Δ_2 such that $l(g) = 0$ satisfies the assumptions of proposition (b). Further, since each such term also satisfies the relation:

$$\text{c-wt}\left(\delta_g, \xi_j\right) \geq \text{c-wt}\left(\theta, \xi_j\right) + p\ \text{c-wt}\left(\rho, \xi_j\right) \geq \text{c-wt}\left(\psi, \xi_j\right) + p - 1\ ,$$

it follows by the second or third inductive hypothesis, according as $\text{c-wt}\left(\delta_g, \xi_j\right)$ is or is not at least $\text{c-wt}\left(E'_M, \xi_j\right)$, that it may be expressed in the required form.

Thirdly, each remaining term $\delta_d\underline{\pi}^{h(d)}$ for d in Δ_3 has at least $\max\left\{2, p^{h-h(d)}\right\}$ distinct subwords equal to words from the set $\left\{\delta_g\underline{\pi}^{l(g)} : g \in \Delta_2\right\}$. For such a term, let k be an integer such that

$0 \leq k \leq h(d)$, and let G be a subword array for $\delta_d \underline{\pi}^k$. Now a subword array F for $\psi\underline{\pi}^{h-h(d)+k}$ is constructed. As for the second type of term described above, the first steps up to the construction of F_k exactly parallel those up to the construction of G_k , and these two families are in one-one correspondence. However, for each of the remaining $h - h(d)$ steps in the construction of $F_{h-h(d)+k}$, all symbols $\underline{\pi}$ are allocated to the S_2 subset and no new star is added. Each copy of a word in G_k has a unique subword equal to δ_d , and corresponds with a collection of $p^{h-h(d)}$ copies in $F_{h-h(d)+k}$ of words, each with a unique subword equal to ψ . To each of these $p^{h-h(d)}$ copies in $F_{h-h(d)+k}$ is allocated a distinct one of the subwords of δ_d equal to a word from the set $\left\{\delta_g \underline{\pi}^{l(g)} : g \in \Delta_2\right\}$. The construction of F is then continued in accordance with this correspondence, by the first or second method described above for each subword ψ in $F_{h-h(d)+k}$ according as the corresponding $\delta_g \underline{\pi}^{l(g)}$ for g in Δ_2 has $l(g) = 1$ or $l(g) = 0$.

As in earlier cases, E is chosen to be the subword array for φ corresponding to the subword array F for $\psi\underline{\pi}^{h-h(d)+k}$ (recall that $h \geq h(d) \geq k \geq 0$, so that $0 \leq h - h(d) + k \leq h$); and again

$$n(G) + h(d) - k \geq n(F) + h - \big(h-h(d)+k\big) \geq nE \ ,$$

and

$$\forall\ \xi \in \Xi\ , \quad \text{c-wt}(G, \xi) \geq \text{c-wt}(F, \xi) \geq \text{c-wt}(E, \xi)\ .$$

Thus each term $\delta_d \underline{\pi}^{h(d)}$ for d in Δ_3 which is of the third type also satisfies the conditions of proposition (b). Further, since each such δ_d has at least two distinct subwords from the set $\left\{\delta_g \underline{\pi}^{l(g)} : g \in \Delta_2\right\}$, it follows that

$$\text{c-wt}\left(\delta_d, \xi_j\right) \geq 2\ \text{c-wt}\left(\psi, \xi_j\right) \geq \text{c-wt}\left(\psi, \xi_j\right) + 1 ;$$

and by the second or third inductive hypothesis, as before, such a term $\delta_d \underline{\pi}^{h(d)}$ may be expressed in the required form.

This completes the proof of the initial case, $k(\rho, \psi) = 1$, for the fourth inductive argument. Next, consider a word $\psi \underline{\pi}^h$ satisfying the conditions described earlier, such that $k(\rho, \psi) > 1$, and assume (fourth inductive hypothesis) that the proposition is established for all words $\psi' \underline{\pi}^{h'}$ such that ψ' has a subword $\rho' \underline{\pi}$ satisfying the conditions $\text{c-wt}\left(\rho', \xi_j\right) \geq 1$ and $k(\rho', \psi') < k(\rho, \psi)$.

Under these conditions, ψ has a subword χ such that $\chi = [\theta, \rho\underline{\pi}]$ (or $\chi = [\rho\underline{\pi}, \theta]$, but again the former will be assumed), and $\text{c-wt}\left(\rho, \xi_j\right) \geq 1$. Let $\omega = \omega\left(\xi_0, \ldots, \xi_{z-1}, \xi_z\right)$ be the word with $\text{c-wt}\left(\omega, \xi_z\right) = 1$ such that

$$\psi \underline{\pi}^h = \omega\left(\xi_0, \ldots, \xi_{z-1}, \chi\right) ,$$

and note that $k\left(\xi_z, \omega\right) = k(\rho, \psi) - 1$.

By law (vi) (b) with $m = 1$,

$$\chi \overset{\text{v}}{=} \prod \left\{\chi_g \underline{\pi}^{l(g)} : g \in \Delta_4\right\}$$

where for each g in Δ_4, χ_g is a cpp-word ending in the symbol $\underline{\gamma}$; and either $l(g) = 1$ and χ_g has at least one subword equal to θ and at least one equal to ρ, or $l(g) = 0$ and χ_g has at least one subword equal to θ and at least p distinct subwords all equal to ρ. By Lemma 1.3.2,

$$\psi \underline{\pi}^h \overset{\text{v}}{=} \prod \left\{\omega\left[\xi_0, \ldots, \xi_{z-1}, \chi_g \underline{\pi}^{l(g)}\right] : g \in \Delta_4\right\} \prod \left\{\delta_d : d \in \Delta_5\right\}$$

where for d in Δ_5 and i in $\underline{z}$,

$$\text{c-wt}\left(\delta_d, \xi_i\right) \geq \text{c-wt}\left(\omega, \xi_i\right) + 2\ \text{c-wt}\left(\chi, \xi_i\right)$$

$$\geq \text{c-wt}\left(\psi\underline{\pi}^h, \xi_i\right) + \text{c-wt}\left(\chi, \xi_i\right) .$$

Each word $\omega\left(\xi_0, \ldots, \xi_{z-1}, \chi_g\underline{\pi}^{l(g)}\right)$ for g in Δ_4 or δ_d for d in Δ_5 may be checked to satisfy the assumptions of proposition (b). That is, if such a word is of the form $\psi'\underline{\pi}^{h'}$ and if $0 \leq k' \leq h'$, then, corresponding to an arbitrary subword array G of $\psi'\underline{\pi}^{k'}$, a subword array F may be constructed for the word $\psi\underline{\pi}^k$ for some appropriate integer k, $0 \leq k \leq h$, and from the hypothesis a corresponding subword array E of φ may be found, which satisfies the hypotheses of proposition (b) also in relation to the array G. Details of the construction of F are omitted; the pattern is similar to that in the initial case.

For those g in Δ_4 such that $l(g) = 1$, the word $\omega\left(\xi_0, \ldots, \xi_{z-1}, \chi_g\underline{\pi}\right)$ may be expressed in the required form, by the fourth inductive hypothesis. All other cpp-words in the final product have higher c-weight in ξ_j then has ψ, and so satisfy either the second or the third inductive hypothesis; and again may be expressed in the required form. This completes the proof of proposition (b), and hence of the lemma. □

Lemmas 1.5.4 and 1.5.5 may be combined to give a more easily manageable result:

1.5.6 COROLLARY. *Corresponding to a group-like variety* $\underline{\underline{V}}$ *and a cpp-word* φ, *there exists an ordered set* Δ *of simple cpp-words such that* $\varphi \overset{\underline{\underline{V}}}{=} \Pi\Delta$ *and* $\varphi \leq' \Pi\Delta$. □

Combining this in turn with Lemma 1.4.2 (a) gives the central result of this chapter:

1.5.7 THEOREM. *Corresponding to a group-like variety* $\underline{\underline{V}}$ *and an arbitrary word* φ *in* B, *there exists an ordered set* Δ *of scpp-words* *such that* $\varphi \overset{\underline{\underline{V}}}{=} \Pi\Delta$ *and* $\varphi \leq' \Pi\Delta$. □

1.5.8 COMMENT AND EXAMPLE

This subsection does not forward the main argument of the thesis, but does point out an application of Lemma 1.5.5 not available through Corollary 1.5.6, which might be useful in calculations in particular groups.

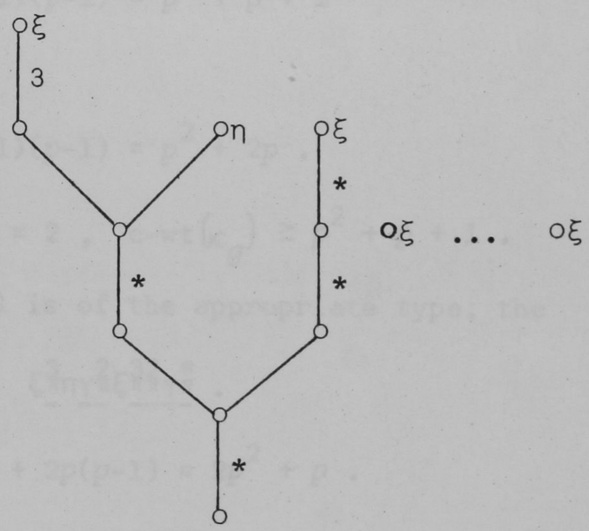
Suppose φ is a cpp-word with c-wt(φ) = w and cpp-wt(φ) = v , and that $\varphi = \varphi'\underline{\pi}^n$ where $n \geq 0$ and the last symbol in φ' is $\underline{\gamma}$. (In this situation, the tree of φ' may be called the *crown* of the tree of φ , and the set of $\underline{\pi}$-arcs linking the vertices labelled φ' and φ may be called its *trunk*. By analogy, the subword φ' itself will be called the *crown* of φ .) For arbitrary group-like variety $\underline{\underline{V}}$, there exists an ordered set of scpp-words whose product is equivalent to φ under the laws of $\underline{\underline{V}}$. One word in this set is obtained from φ simply by moving all symbols $\underline{\pi}$ from their positions in φ to the end of the word. It is clear that this is the only word in the set whose c-weight is equal to that of φ : modulo words of higher c-weight, Law (vi) shows that a symbol $\underline{\pi}$ may be moved to any position where it still makes a word. If φ was not already an scpp-word, then the cpp-weight of the word obtained in this way is stricly greater than that of φ ; often it is substantially greater. On the other hand, there are words in the set whose cpp-weight is equal to that of φ ; each of these has c-weight greater than or equal to the cpp-weight of the crown of φ . The former word corresponds to the subword array E_m for φ which contains a single labelling of φ in which every symbol $\underline{\pi}$ is starred; words of the latter type correspond to subword arrays of $\varphi'\underline{\pi}^{n'}$ for $0 \leq n' \leq n$ in which no symbol $\underline{\pi}$ is starred.

The following example is included as an illustration of the use of Lemma 1.5.5 in obtaining information about the minimum weights of commutators κ_g corresponding to each integer $h(g)$ in expressions

$\varphi \overset{\vee}{=} \prod\left\{\kappa_g \underline{\pi}^{h(g)} : g \in \Gamma\right\}$. It is repeated that this information is not of any use in developing further results in this thesis. Note that among subword arrays E_i for φ with a given number $n(E_i)$ of stars, that with least value of c-wt(E_i) occurs only if the path from every starred $\underline{\pi}$-edge in a tree to the root of the tree does not pass through an unstarred $\underline{\pi}$-edge.

Let $\varphi = \xi\underline{\pi}\eta\underline{\gamma\pi}\xi\underline{\pi\pi\gamma\pi}$; then c-wt$(\varphi) = 3$, and cpp-wt$(\varphi) = 2p^3 + p^2$. Every subword array E for φ has c-wt(E) between those bounds. Let $E_m = \{\xi\underline{\overset{*}{\pi}}\eta\underline{\gamma\overset{*}{\pi}}\xi\underline{\overset{*}{\pi}\overset{*}{\pi}\gamma\overset{*}{\pi}}\}$; this is the only subword array for φ whose c-weight is equal to 3 , and it satisfies $n(E_m) = 5$. If $\varphi \overset{\vee}{=} \prod\left\{\kappa_g \underline{\pi}^{h(g)} : g \in \Gamma\right\}$, every g in Γ such that $h(g) \geq 5$ has c-wt$(\kappa_g) \geq 3$.

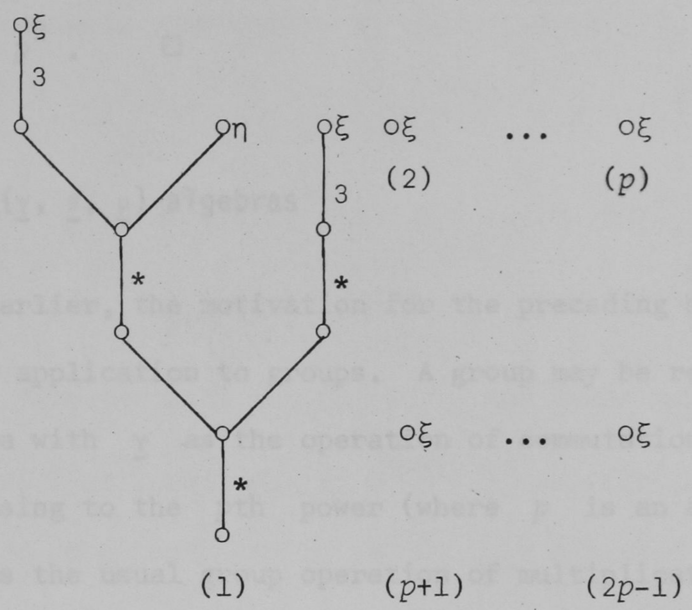
Consider subword array E_g for φ such that $n(E_g) = 4$. Two of these are relevant, and they are not essentially different; one is illustrated opposite. For each such g in Γ , $h(g) = 4$ and c-wt$(E_g) \geq 3 + p - 1 = p + 2$, so that c-wt$(\kappa_g) \geq p + 2$.



(1) (2) ... (p)

Among arrays E_g for φ such that $n(E_g) = 3$, it is easy to see that the one with the least value of c-wt(E_g) is that illustrated below. For each such g in Γ , $h(g) = 3$, and

$$\text{c-wt}(\kappa_g) \geq \text{c-wt}(E_g) \geq 3 + 2(p-1) = 2p + 1 .$$



Two arrays, E_g and E_h say, with $n(E_g) = n(E_h) = 2$ are considered. In E_g and E_h the basic labellings of φ which occur are $\xi\underline{\pi^{3}}\eta\underline{\gamma\pi^{*}}\xi\underline{\pi^{3}\pi^{2}\gamma\pi^{*}}$ and $\xi\underline{\pi^{3}}\eta\underline{\gamma\pi^{2}}\xi\underline{\pi^{3}\pi^{*}\gamma\pi^{*}}$ respectively; diagrams are left for the reader to supply. It can be seen that

$$\text{c-wt}(E_g) = 3 + 1(p-1) + (p+1)(p-1) = p^2 + p + 1$$

and

$$\text{c-wt}(E_h) = 3 + 2(p-1) + (p+1)(p-1) = p^2 + 2p \,.$$

Hence for each g in Γ such that $h(g) = 2$, $\text{c-wt}(\kappa_g) \geq p^2 + p + 1$.

Only one array E_g such that $n(E_g) = 1$ is of the appropriate type; the basic labelling of φ contained in it is $\xi\underline{\pi^{3}}\eta\underline{\gamma\pi^{2}}\xi\underline{\pi^{3}\pi^{2}\gamma\pi^{*}}$.

$$\text{c-wt}(E_g) = 3 + (p-1) + 2(p-1) + 2p(p-1) = 2p^2 + p \,.$$

Hence for each g in Γ such that $h(g) = 1$, $\text{c-wt}(\kappa_g) \geq 2p^2 + p$. Note that $2p^2 + p$ is the cpp-weight of the crown of φ; and if $\text{c-wt}(\kappa_g) = 2p^2 + p$ and $h(g) = 1$ then $\text{cpp-wt}\left(\kappa_g \underline{\pi}^{h(g)}\right) = \text{cpp-wt}(\varphi)$.

Finally, if E_g is the array E_M which contains no stars, then $\text{c-wt}(E_g) = 2p^3 + p^2 = \text{cpp-wt}(\varphi)$. For each g in Γ_g such that $h(g) = 0$, c-wt$\left(\kappa_g\right) \geq 2p^3 + p^2$. □

1.6 Groups as $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras

As stated earlier, the motivation for the preceding sections of this chapter is their application to groups. A group may be regarded as a $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra with $\underline{\gamma}$ as the operation of commutation, $\underline{\pi}$ as the operation of raising to the pth power (where p is an arbitrary but fixed prime) and $\underline{\mu}$ is the usual group operation of multiplication. The notes accompanying the definitions in 1.3.1 make it clear that groups satisfy laws (i) to (iii). The main result of this section is the proof, in Theorem 1.6.2, that the variety of all groups is a group-like variety of $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras. Some results from other sources, used in this proof, are collected for convenience as Lemma 1.6.1. Corollary 1.6.3, which will be used repeatedly in later chapters, restates the main result of the preceding section in terms of groups.

1.6.1 LEMMA. *Let* α, β , *and* γ *be elements of a group* G , *and* h *be a positive integer. Then*

(a) (see, for example, Huppert [11], Kapitel III, Hilfsatz 1.2 (p. 253), or Hanna Neumann [19], 33.34 (1), (p. 85)).

$$[\alpha\beta, \gamma] = [\alpha, \gamma]^{\beta}[\beta, \gamma]$$

and

$$[\alpha, \beta\gamma] = [\alpha, \gamma][\alpha, \beta]^{\gamma} .$$

(b) (*cf*. P. Hall, [9], Theorem 3.2).

$$(\alpha\beta)\underline{\pi}^{h} = \alpha\underline{\pi}^{h}\beta\underline{\pi}^{h} \prod \left\{\kappa_{g}\underline{\pi}^{h(g)} : g \in \Gamma_{1}\right\}$$

where for g *in* Γ_1 , $0 \leq h(g) \leq h$ *and* κ_g *is a commutator in* G *with at least* $p^{h-h(g)}$ *entries from the set* $\{\alpha, \beta\}$.

(c) (W. Haebich, personal communication. A related but more detailed

result in terms of basic commutators is contained in [7], Lemma 3.4.6.)

$$[\alpha\underline{\pi}^h, \beta] = [\alpha, \beta]\underline{\pi}^h \prod\left\{\kappa_g\underline{\pi}^{h(g)} : g \in \Gamma_2\right\},$$

$$[\alpha, \beta\underline{\pi}^h] = \prod\left\{\kappa_g\underline{\pi}^{h(g)} : g \in \Gamma_3\right\}[\alpha, \beta]\underline{\pi}^h$$

where for g *in* Γ_2 *,* $0 \le h(g) \le h$ *and* κ_g *is a commutator with at least* $p^{h-h(g)}$ *entries from the set* $\{\alpha, [\alpha, \beta]\}$ *, and hence may also be expressed as a commutator with at least* $\max\{2, p^{h-h(g)}\}$ *entries equal to* α *and at least one equal to* β *; and for* g *in* Γ_3 *,* $0 \le h(g) \le h$ *and* κ_g *may be expressed as a commutator with at least* $\max\{2, p^{h-h(g)}\}$ *entries equal to* β *and at least one equal to* α *.*

Proof. Statement (a) is so familiar as to require no proof. It is easily verified by expanding both sides.

Statement (b) omits one of the premises required by Hall in [9], that the group G should be nilpotent. However the factor group $G/\gamma_{p^h}(G)$, where $\gamma_{p^h}(G)$ is the p^hth term of the lower central series of G (discussed in more detail in Section 1.7), is clearly nilpotent; and it is easy to see that if the required result is true for the factor group then it is true for G , since an arbitrary element of $\gamma_{p^h}(G)$ is, by definition, equal to a product of commutators of weight at least p^h .

(In terms of Hall's proof, the condition of nilpotency is required to ensure that his "commutator collecting process" terminates after finitely many steps. If this process is continued until all commutators of weight less than or equal to $p^h - 1$ are collected (a finite process) then the expression remaining, though not in "collected form", satisfies the requirements of the present lemma.)

Another point to be noted is that the terms of the final product are claimed to be of the form $\kappa_{\underline{g}}\pi^{h(g)} = \kappa_g^{p^{h(g)}}$ where Hall gives only $\kappa_g^{e(g)}$ where $p^{h(g)} | e(g)$. This change may be achieved simply by rewriting $\kappa_g^{e(g)}$ as a product of factors each equal to $\kappa_{\underline{g}}\pi^{h(g)}$ or its inverse.

To prove the first part of statement (c), note that

$$\begin{aligned} [\alpha\underline{\pi}^h, \beta] &= \alpha^{-p^h} (\alpha^{p^h})^\beta \\ &= \alpha^{-p^h} (\alpha^\beta)^{p^h} \\ &= \alpha^{-p^h} (\alpha[\alpha, \beta])^{p^h} , \end{aligned}$$

and then use part (b) to show that

$$(\alpha[\alpha, \beta])^{p^h} = \alpha^{p^h} [\alpha, \beta]^{p^h} \prod \left\{\kappa_{\underline{g}}\pi^{h(g)} : g \in \Gamma_2\right\} ,$$

which gives an expression of the required form.

The second part of statement (c) follows from the first by the observation that for all ξ and η in G , $[\xi, \eta] = [\eta, \xi]^{-1}$. □

1.6.2 THEOREM. *Every variety of groups is a group-like variety of* $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$*-algebras.*

Proof. What remains to be proved, following the comment in the introduction to this chapter, is that laws (iv), (v) and (vi) hold in an arbitrary group G .

To show that law (iv) holds in G , let $\{\alpha_i : i \in \underline{m}\}$ and $\{\beta_j : j \in \underline{n}\}$ be non-empty sets of elements of G , and proceed by induction on $m + n$ to show that

$$\left[\prod \{\alpha_i : i \in \underline{m}\}, \prod \{\beta_j : j \in \underline{n}\}\right]$$
$$= \prod \{[\alpha_i, \beta_j] : (i, j) \in \underline{m} \times \underline{n}\} \prod \{\delta_d : d \in \Delta_1\}$$

where for each d in Δ_1 there exists a triple (i, j, k) either in

$\underline{m} \times \underline{n} \times \underline{n}$ with $j \neq k$ such that $[\alpha_i, \beta_j, \beta_k] \leq' \delta_d$, or in $\underline{m} \times \underline{n} \times \underline{m}$ with $i \neq k$ such that $[\alpha_i, \beta_j, \alpha_k] \leq' \delta_d$, where of course the relation $\leq'$ is that described in 1.2.3.

When $m + n = 2$, the least possible value, the result is trivially true. When $m + n > 2$, either $m \geq 2$ or $n \geq 2$; suppose the former. From Lemma 1.6.1 (a) and then by the inductive hypothesis,

$$\left[\prod \{\alpha_i : i \in \underline{m}\}, \prod \{\beta_j : j \in \underline{n}\}\right]$$

$$= \left[\prod \{\alpha_i : i \in \underline{m-1}\}, \prod \{\beta_j : j \in \underline{n}\}\right]^{\alpha_{m-1}} \left[\alpha_{m-1}, \prod \{\beta_j : j \in \underline{n}\}\right]$$

$$= \left(\prod \{[\alpha_i, \beta_j] : (i, j) \in \underline{m-1} \times \underline{n}\} \prod \{\delta_g : g \in \Gamma_1\}\right)^{\alpha_{m-1}}$$

$$\prod \{[\alpha_{m-1}, \beta_j] : j \in \underline{n}\} \prod \{\delta_g : g \in \Gamma_2\},$$

where for each g in Γ_1 or Γ_2, δ_g satisfies the conditions required of elements of the set $\{\delta_d : d \in \Delta_1\}$. Since

$$\delta_g^{\alpha_{m-1}} = \delta_g [\delta_g, \alpha_{m-1}]$$

and

$$[\alpha_i, \beta_j]^{\alpha_{m-1}} = [\alpha_i, \beta_j][\alpha_i, \beta_j, \alpha_{m-1}],$$

and since all new commutators introduced by rearranging the order of the factors in the product above also satisfy the conditions required of elements of the set $\{\delta_d : d \in \Delta_1\}$, it is readily seen that the expression above is equal to

$$\prod \{[\alpha_i, \beta_j] : (i, j) \in \underline{m} \times \underline{n}\} \prod \{\delta_d : d \in \Delta_1\},$$

as required.

To show that law (v) holds in G, let $\{\alpha_i : i \in \underline{m}\}$ be an arbitrary non-empty set of elements of G. Consider the proposition $P(w)$:

Let $\{\delta_d : d \in \underline{l}\}$ be a set of commutators, each with at least w entries from the set $\{\alpha_i : i \in \underline{m}\}$, and let k be the least non-negative integer such that $w \geq p^{h-k}$; then

$$\left(\prod\{\delta_d : d \in \underline{l}\}\right)\underline{\pi}^k = \prod\left\{\kappa_g\underline{\pi}^{h(g)} : g \in \Gamma_3\right\}$$

where for g in Γ_3, $0 \leq h(g) \leq k \leq h$, the commutator κ_g has at least $p^{h-h(g)}$ entries from the set $\{\alpha_i : i \in \underline{m}\}$, there is a subset Γ_3^* of Γ_3 such that

$$\left\{\delta_d\underline{\pi}^k : d \in \underline{l}\right\} = \left\{\kappa_g\underline{\pi}^{h(g)} \ : g \in \Gamma_3^*\right\},$$

and for g in $\Gamma_3\backslash\Gamma_3^*$, the commutator κ_g has at least two entries from the set $\{\alpha_i : i \in \underline{m}\}$.

The proposition $P(1)$ is the required result, that law (v) holds in G. The proposition $P(p^h)$ is clearly true, since in that proposition each κ_g may be taken to be one of the δ_d, and each $h(g)$ to be zero. For arbitrary w less than p^h, suppose that $P(v)$ is true for $v > w$.

Now proceed by a second induction, on l. When $l = 1$, the result is trivial. Suppose $l > 1$, and the result established for expressions with fewer than l factors. Lemma 1.6.1 (b) shows that

$$\left(\prod\{\delta_d : d \in \underline{l-1}\}.\delta_{l-1}\right)\underline{\pi}^k$$

$$= \left(\prod\{\delta_d : d \in \underline{l-1}\}\right)\underline{\pi}^k . \delta_{l-1}\underline{\pi}^k . \prod\left\{\delta_g\underline{\pi}^{k(g)} : g \in \Gamma_4\right\}$$

where for g in Γ_4, δ_g is a commutator with at least $\min\{2, p^{k-k(g)}\}$ entries equal either to the product $\prod\{\delta_d : d \in \underline{l-1}\}$ or to the element δ_{l-1}. By law (iv) applied as often as required, each such δ_g for g in Γ_4 may be expressed as a product of commutators, each with at least

$\min\{2, p^{k-k(d)}\}$ entries from the set $\{\delta_d : d \in \underline{l-1}\}$, and hence as a product of commutators each with at least $\min\{2w, p^{h-k(d)}\}$ entries from the set $\{\alpha_i : i \in \underline{m}\}$. The inductive hypothesis on w shows that each of these may be expressed in the required form. From the inductive hypothesis on l, the expression $\left(\prod \{\delta_d : d \in \underline{l-1}\}\right)_{\underline{\pi}}^{k}$ may be expressed as the product of a set of powers of commutators of the required form which contains the subset $\left\{\delta_{d\underline{\pi}}^{k} : d \in \underline{l-1}\right\}$. This, together with the factor $\delta_{l-1\underline{\pi}}^{k}$ gives the distinguished subset required in the total product; and so the truth of $P(w)$ is proved.

By induction the truth of $P(1)$ follows, and law (v) holds in G.

That law (vi) holds in G is already shown in the proof of Lemma 1.6.1 (c). □

Theorems 1.5.7 and 1.6.2 combine to give a result which will be used repeatedly in later chapters.

1.6.3 COROLLARY. *Corresponding to an arbitrary element* ρ *of a group* G *there exists a set* Δ *of scpp-elements of* G, *such that* $\rho = \Pi\Delta$ *and, in the sense of* 1.2.3, $\rho \leq' \Pi\Delta$. □

1.7 Some descending central series of groups

The series described in sub-sections 1.7.1, 1.7.2 and 1.7.3 are made up of weight ideals of the group in which they occur. Each may alternatively be characterised as the "lowest" descending series of a group to have a particular property. Each consists of verbal subgroups corresponding to specified, quite small sets of words. Finally, the possible "stationary" behaviour of each is examined. In sub-section 1.7.4, the "refined cpp-series", a refinement of the series described in 1.7.3, is described. Though its terms are not, strictly speaking, weight ideals of the group, each is the subgroup generated by two weight ideals and is verbal. In subsection 1.7.5, cpp-nilpotent groups and their different "classes" are briefly discussed.

Some preliminary definitions and notation are required. Firstly, if H and K are subgroups of a group, then

$$[H, K] = \mathrm{sgp}\langle [\rho, \sigma] : \rho \in H, \sigma \in K\rangle .$$

Denote by $\mathcal{H}$ a series

$$G = H_1 \geq H_2 \geq \dots \geq H_i \geq \dots$$

of subgroups of a group G. Such a series is called *central* if for all i in $\mathbb{Z}^+$,

$$[H_i, H_1] \subseteq H_{i+1} .$$

It is *strongly central* if for all i and j in $\mathbb{Z}^+$,

$$[H_i, H_j] \subseteq H_{i+j} .$$

Clearly every strongly central series is central, but examples of central series which are not strongly central are easy to construct. For example, if $\mathcal{H}$ is a central series such that $[H_2, H_2]$ is not contained in H_5, and if a series $\mathcal{K}$ is defined by:

$$H_1 = K_1 = K_2 \quad \text{and for } i \geq 1 , \quad H_i = K_{i+1} ,$$

then the series $\mathcal{K}$ is central, but since $[K_3, K_3]$, which equals $[H_2, H_2]$, is not contained in $K_6 = H_5$, the series is not strongly central.

A central series (or indeed, a series in which each factor H_i/H_{i+1} is abelian) is called *elementary* if for some fixed prime p and for all i in $\mathbb{Z}^+$,

$$H_i^p \subseteq H_{i+1} .$$

It is called *restricted elementary* if for all i in $\mathbb{Z}^+$,

$$H_i^p \subseteq H_{ip} . \quad \square$$

1.7.1 THE LOWER CENTRAL SERIES

Define for the $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-word algebra B, and i in Z^+,

$$\gamma_i(B) = \left\{\varphi \in B : \text{c-wt}(\varphi) = w^1_{1,0}(\varphi) \geq i\right\}.$$

For an arbitrary group G and surjective homomorphism $\underline{\alpha} : B \to G$, define

$$\gamma_i(G) = \gamma_i(B)\underline{\alpha}.$$

Lemma 1.2.5 shows that $\gamma_i(G)$ is well-defined, independently of the particular surjective homomorphism $\underline{\alpha}$. Denote by $\mathcal{G}$ the series:

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots \geq \gamma_i(G) \geq \ldots .$$

It will be shown below that $\mathcal{G}$ is the well-known lower central series of G. From the definition above and Lemma 1.4.3 in conjunction with Theorem 1.6.2, it is clear that $\gamma_i(G)$ is generated as a subgroup of G by the set of all homomorphic images in B of c-words in B whose c-weight is at least i. Property (b) below gives a sharper result than this. The series $\mathcal{G}$ has the following properties, listed here mainly for purposes of comparison with the corresponding properties in 1.7.3 and 1.7.4:

(a) $\mathcal{G}$ is strongly central.

(b) The subgroup $\gamma_i(G)$ is generated by the set of homomorphic images in G of the single left-normed commutator word $[\xi_0, \ldots, \xi_{i-1}]$.

(c) The series $\mathcal{G}$ is the "lowest" descending central series, in the sense that if $\mathcal{H}$ is a descending central series of G, then for all i in Z^+,

$$\gamma_i(G) \subseteq H_i.$$

(d) For all i in Z^+, $\gamma_{i+1}(G) = [\gamma_i(G), G]$.

(e) If $\gamma_i(G) = \gamma_{i+1}(G)$, then for all j in Z^+,

$$\gamma_i(G) = \gamma_{i+j}(G).$$

Proof. That the lower central series of a group has properties (a) to (e) is well-known. For example, they are given by Huppert in [12] as follows:

(a) Hauptsatz 2.11 (b), p. 265.

(b) Definition 1.9, pp. 256-257 (used by Huppert as the definition of the lower central series).

(c) (without statement) in the proof (a) ⇒ (b) of Hauptsatz 2.3, p. 260.

(d) stated at the end of 1.9, p. 257, as a consequence of Hilfsatz 1.8.

(e) is a special case $(n = 1)$ of Satz 2.13 (b), p. 266, or of course is easily deduced from (d).

It remains to be shown that the series G defined above is in fact the lower central series as defined by Huppert. The latter is, for this proof, denoted

$$G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_i \supseteq \ldots .$$

From its definition, G is strongly central; so if it can be shown that for all i in Z^+, $\gamma_i(G) \subseteq G_i$, then the identity of the two will be established, since the reverse inclusion follows from property (c) of the lower central series. Proceed by induction on i. By definition, $G = G_1 = \gamma_1(G)$. Suppose $i > 1$. As noted earlier, $\gamma_i(G)$ is generated by the images under a surjective homomorphism $\underline{\alpha} : B \to G$ of c-words of c-weight at least i. If χ is such a word, then $\chi = [\varphi, \psi]$ where φ and ψ are c-words, c-wt$(\varphi) = j$, c-wt$(\psi) = k$, and $j + k \geq i$. If either j or k is greater than or equal to i, then consider the corresponding word φ or ψ; eventually the case may be reduced to one with $j < i$, $k < i$, and $j + k \geq i$. By induction, $\varphi\underline{\alpha} \in G_j$ and $\psi\underline{\alpha} \in G_k$; so, since the lower central series also is strongly central,

$$\chi\underline{\alpha} = [\varphi\underline{\alpha}, \psi\underline{\alpha}] \in [G_j, G_k]$$
$$\subseteq G_{j+k} \subseteq G_i .$$

Since $\gamma_i(G)$ is generated by such elements $\chi\underline{\alpha}$, the result that

$\gamma_i(G) \subseteq G_i$ follows. Hence $\gamma_i(G) = G_i$, as required. □

1.7.2 THE LOWER ELEMENTARY CENTRAL SERIES

Define, for the $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-word algebra B and a fixed prime p used in defining weights on it and in interpreting the operation $\underline{\pi}$ in groups,

$$\varepsilon_i(B) = \left\{\varphi \in B : w^1_{1,1}(\varphi) \geq i\right\} .$$

For an arbitrary group G and a surjective homomorphism $\underline{\alpha} : B \to G$, define

$$\varepsilon_i(G) = \varepsilon_i(B)\underline{\alpha} .$$

Again, Lemma 1.2.5 shows that $\varepsilon_i(G)$ is well-defined. Denote by E the series

$$G = \varepsilon_1(G) \supseteq \varepsilon_2(G) \supseteq \ldots \supseteq \varepsilon_i(G) \supseteq \ldots ,$$

which will be called the *lower elementary central series* of G. The name is justified by properties (a) and (c) below. From Theorem 1.5.7, it is clear that $\varepsilon_i(G)$ is generated as a subgroup by the set of all homomorphic images in G of words in B of the form $\kappa\underline{\pi}^k$ where κ is a c-word of c-weight at least $i - k$. Again, property (b) below gives a sharper statement than this. The series E has the following properties:

(a) E is strongly central and elementary.

(b) The subgroup $\varepsilon_i(G)$ is generated by the set of homomorphic images in G of the i words in the set

$$\left\{[\xi_0, \ldots, \xi_{i-k-1}]\underline{\pi}^k : 0 \leq k \leq i-1\right\} .$$

(c) E is the "lowest" descending elementary central series of G, in the sense that if H is an elementary central series then for all i in Z^+,

$$\varepsilon_i(G) \subseteq H_i .$$

(d) For all i in Z^+, $\varepsilon_{i+1}(G) = \mathrm{sgp}\left\langle [\varepsilon_i(G), G], (\varepsilon_i(G))^p \right\rangle$.

(e) If $\varepsilon_i(G) = \varepsilon_{i+1}(G)$, then for all j in Z^+ ,

$$\varepsilon_i(G) = \varepsilon_{i+j}(G) .$$

Proof. (a) To show that E is strongly central, suppose $\rho \in \varepsilon_i(G)$ $\sigma \in \varepsilon_j(G)$ for some i and j in Z^+ . Then there exist words φ in $\varepsilon_i(B)$ and ψ in $\varepsilon_j(B)$ and a surjective homomorphism $\underline{\alpha} : B \to G$ such that $\varphi\underline{\alpha} = \rho$ and $\psi\underline{\alpha} = \sigma$. By definition, $[\varphi, \psi] \in \varepsilon_{i+j}(B)$, whence

$$[\rho, \sigma] = [\varphi, \psi]\underline{\alpha} \in \varepsilon_{i+j}(G) ,$$

as required. To show that E is elementary, note that

$$\rho\underline{\pi} = (\varphi\underline{\pi})\underline{\alpha} \in \varepsilon_{i+1}(G) ,$$

as required.

(b) For an integer k , $0 \le k \le i-1$, let $P(k)$ be the proposition: corresponding to each c-word φ in B of c-weight $i - k$, there exist ordered sets $\{\psi_g : g \in \Gamma\}$ of left-normed c-words of B and $\{k(g) : g \in \Gamma\}$ of non-negative integers, and a homomorphism $\underline{\alpha} : B \to G$, such that

$$\left(\varphi\underline{\pi}^k\right)\underline{\alpha} = \left(\prod\left\{\psi_g\underline{\pi}^{k(g)} : g \in \Gamma\right\}\right)\underline{\alpha}$$

and for each g in Γ ,

$$\text{c-wt}\left(\psi_g\right) = i - k(g) \ge i - k .$$

Since it has already been pointed out that $\varepsilon_i(G)$ is generated by elements satisfying the conditions on $\varphi\underline{\pi}^k$ for $0 \le k \le i-1$, what is required is a proof that $P(k)$ holds for $0 \le k \le i-1$.

Clearly $P(0)$ is true; in fact it is 1.7.1 (b). Suppose $k > 0$, and propositions $P(j)$ for $0 \le j \le k-1$ all true. Let φ be a c-word in B of weight $i - k$. By 1.7.1 (b), there exists a set $\{\psi_g : g \in \Gamma_1\}$ of left-normed commutator words in B of c-weight $i - k$ and a homomorphism

$\underline{\alpha} : B \to G$ such that $\varphi\underline{\alpha} = \left(\prod \{\psi_g : g \in \Gamma_1\}\right)\underline{\alpha}$. Since G belongs to a grouplike variety, law (v) shows that

$$(\varphi\underline{\pi}^k)\underline{\alpha} = \left(\prod \{\psi_g : g \in \Gamma_1\}\right)\underline{\pi}^k\underline{\alpha} = \left(\prod \left\{\chi_{d\underline{\pi}^{k(d)}} : d \in \Gamma_2\right\}\right)\underline{\alpha}$$

where $\left\{\chi_{d\underline{\pi}^{k(d)}} : d \in \Gamma_2\right\}$ has a subset equal to $\left\{\psi_{g\underline{\pi}^k} : g \in \Gamma_1\right\}$, and where for each d in Γ_2 corresponding to an element outside this subset, $0 \le k(d) \le k$ and χ_d is a c-word with at least $\max\{2, p^{k-k(d)}\}$ distinct subwords from the set $\{\psi_g : g \in \Gamma_1\}$, whence

$$\text{c-wt}(\chi_d) \ge \max\{2(i-k), p^{k-k(d)}(i-k)\} .$$

Note that $p^{k-k(d)}(i-k) \ge i - k(d)$, since

$$p^{k-k(d)} \ge 1 + (k-k(d)) \ge \frac{i-k+k-k(d)}{i-k} .$$

The words $\psi_{g\underline{\pi}^k}$ for g in Γ_1 are of the required form, and the other words $\chi_{d\underline{\pi}^{k(d)}}$ all satisfy the conditions of the inductive hypothesis on k , and so may be expressed as products of words of the required form.

Hence, by induction, $P(k)$ is true for $0 \le k \le i-1$, and thus property (b) holds.

(c) Let $G = H_1 \supseteq H_2 \supseteq \ldots \supseteq H_i \supseteq \ldots$ be an elementary central series. By definition, $G = H_1 = \varepsilon_1(G)$. Suppose $i > 1$, and assume inductively that $\varepsilon_j(G) \subseteq H_j$ for $1 \le j \le i-1$. From (b), $\varepsilon_i(G)$ is generated by elements of the form $(\varphi\underline{\pi}^k)\underline{\alpha}$ where $\underline{\alpha}$ is a homomorphism from B to G and φ is a c-word in B of c-weight $i - k$. If $k \ge 1$, then by the inductive hypothesis

$$(\varphi\underline{\pi}^{k-1})\underline{\alpha} \in \varepsilon_{i-1}(G) \subseteq H_{i-1} ,$$

whence

$$\left(\varphi\underline{\pi}^k\right)\underline{\alpha} = \left(\varphi\underline{\pi}^{k-1}\right)\underline{\alpha\pi} \in H_{i-1}\underline{\pi} \subseteq H_i ,$$

as required. If $k = 0$, then $\varphi \in \gamma_i(G) \subseteq H_i$, by property (c) of 1.7.1.

(d) From (b), $\varepsilon_{i+1}(G)$ is generated by elements of the form $\left(\varphi\underline{\pi}^k\right)\underline{\alpha}$ where φ is a left-normed c-word of c-weight $i + 1 - k$ in B , and $\underline{\alpha}$ is a homomorphism from B to G . If $k \geq 1$, then $\left(\varphi\underline{\pi}^{k-1}\right)\underline{\alpha} \in \varepsilon_i(G)$, and

$$\varphi\underline{\pi}^k\underline{\alpha} = \varphi\underline{\pi}^{k-1}\underline{\alpha\pi} \in \left(\varepsilon_i(G)\right)^p .$$

If $k = 0$, then since φ is left-normed, $\varphi = \left[\varphi_1, \varphi_2\right]$ where $\varphi_1 \in \varepsilon_i(G)$ and $\varphi_2 \in G$, so that

$$\varphi\underline{\alpha} = \left[\varphi_1\underline{\alpha}, \varphi_2\underline{\alpha}\right] \in \left[\varepsilon_i(G), G\right] .$$

Hence

$$\varepsilon_{i+1}(G) \subseteq \operatorname{sgp}\left\langle \left|\varepsilon_i(G), G\right|, \left(\varepsilon_i(G)\right)^p \right\rangle .$$

The reverse inclusion is immediate from the definitions.

(e) Proved by induction on j . For $j \geq 2$, suppose $\varepsilon_{i+j-2}(G) = \varepsilon_{i+j-1}(G)$. Then

$$\begin{aligned} \varepsilon_{i+j}(G) &= \left[\varepsilon_{i+j-1}(G), G\right]\left(\varepsilon_{i+j-1}(G)\right)^p && \text{by (d)} \\ &= \left[\varepsilon_{i+j-2}(G), G\right]\left(\varepsilon_{i+j-2}(G)\right)^p && \text{by hypothesis} \\ &= \varepsilon_{i+j-1}(G) && \text{by (d),} \end{aligned}$$

as required. □

1.7.3 THE LOWER RESTRICTED ELEMENTARY CENTRAL SERIES

Define, for the $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-word algebra B , and the fixed prime involved in defining weights and interpreting the operation $\underline{\pi}$ in groups,

$$\pi_i(B) = \left\{\varphi \in B : \text{cpp-wt}(\varphi) = w^p_{1,0}(\varphi) \geq i\right\}.$$

For arbitrary group G and surjective homomorphism $\underline{\alpha} : B \to G$, define $\pi_i(G) = \pi_i(B)\underline{\alpha}$. As before, Lemma 1.2.5 shows that $\pi_i(G)$ is well-defined. Denote by P the series

$$G = \pi_1(G) \supseteq \pi_2(G) \supseteq \ldots \supseteq \pi_i(G) \supseteq \ldots$$

which, to save repetition of the long name used as a heading to this section, will more usually be called the *cpp-series* of G. From Theorem 1.5.7, it is clear that $\pi_i(G)$ is generated as a subgroup by the set of all homomorphic images in G of scpp-words in B whose cpp-weight is at least i; as before, a sharper statement is given as property (b).

The series P has the properties:

(a) P is strongly central and is restricted elementary.

(b) For i in Z^+, let l be the integer such that

$$p^{l-1} < i \leq p^l,$$

and for $0 \leq k \leq l$, let $w(k)$ be the least integer such that $p^k w(k) \geq i$. Then $\pi_i(G)$ is the subgroup generated by the set of homomorphic images in G of the $l + 1$ words in the set

$$\left\{[\xi_0, \ldots, \xi_{w(k)-1}]\underline{\pi}^k : 0 \leq k \leq l\right\}.$$

(c) The series P is the "lowest" descending restricted elementary central series, in the sense that if H is restricted elementary and central, then for all i in Z^+, $\pi_i(G) \subseteq H_i$.

(d) For all i in Z^+,

$$\pi_{i+1}(G) = \text{sgp}\left\langle[\pi_i(G), G], (\pi_{r(i)}(G))^p\right\rangle$$

where $r(i) = [\,i\ /p] + 1$.

(e) If $\pi_i(G) = \pi_{ip}(G)$, then $\pi_i(G) = \pi_{i+j}(G)$ for all j in Z^+.

Proof. (a) The proof is routine, similar to that of 1.7.2 (a), and is omitted.

(b) As noted earlier, $\pi_i(G)$ is generated by the set of all homomorphic images in G of scpp-words whose cpp-weights are at least i ; that is, of words of the form $\varphi\underline{\pi}^k$ where, for $0 \le k \le l$, φ is a c-word of c-weight at least $w(k)$, and where, for $k > l$, φ is a c-word of c-weight at least 1 . If $k \ge l$, then $\varphi\underline{\pi}^k$ is a homomorphic image of $\xi_0\underline{\pi}^k$, so in this case the result clearly holds.

Let $R(w)$ be the proposition: Corresponding to each c-word φ of c-weight at least w , let k be the least integer such that $w(k) \le w$ (that is, such that $wp^k \ge i$); then there exist sets $\{\psi_g : g \in \Gamma\}$ of left-normed c-words in B and $\{k_g : g \in \Gamma\}$ of non-negative integers, and a homomorphism $\underline{\alpha} : B \to G$ such that

$$(\varphi\underline{\pi}^k)\underline{\alpha} = \left(\prod \left\{\psi_g\underline{\pi}^{k(g)} : g \in \Gamma\right\}\right)\underline{\alpha}$$

and for each g in Γ ,

$$k(g) \le k \quad \text{and} \quad \text{c-wt}(\psi_g) \ge w(k(g)) \ .$$

Since $w(0) = i$, the propositions $R(j)$ for $j \ge i$ follow from 1.7.1 (b). The required result will be established if it is shown that $R(w)$ is true for $1 \le w < i$. Suppose that $w < i$, and, inductively, that $R(x)$ is true for all x such that $x > w$. Let φ be a c-word in B of c-weight w ; then, by 1.7.1 (b) there exist a set $\{\psi_g : g \in \Gamma_1\}$ of left-normed commutator words in B of c-weight precisely w and a homomorphism $\underline{\alpha} : B \to G$ such that

$$\varphi\underline{\alpha} = \left(\prod \{\psi_g : g \in \Gamma_1\}\right)\underline{\alpha} \ .$$

Since G belongs to a group-like variety, law (v) shows that

$$\begin{aligned}\varphi\underline{\pi}^k\underline{\alpha} &= \left(\prod \{\psi_g : g \in \Gamma_1\}\right)\underline{\pi}^k\underline{\alpha} \\ &= \left(\prod \left\{\chi_d\underline{\pi}^{k(d)} : d \in \Gamma_2\right\}\right)\underline{\alpha}\end{aligned}$$

where $\left\{\chi_{\underline{d}}\pi^{k(d)} : d \in \Gamma_2\right\}$ has a subset equal to $\left\{\psi_{\underline{g}}\pi^k : g \in \Gamma_1\right\}$, and where for each d in Γ_2 corresponding to an element outside this subset, $0 \le k(d) \le k$, and χ_d is the image in B of a c-word with c-weight at least $\max\{2, p^{k-k(d)}\}$ under an endomorphism mapping each initial subword of that c-word to an element of the set $\{\psi_g : g \in \Gamma_1\}$; whence

$$\text{c-wt}\left(\chi_d\right) \ge \max\{2w, p^{k-k(d)}w\} .$$

The words $\chi_{\underline{d}}\pi^{k(d)}$ for d in Γ_2 which are in the distinguished subset are of the required form; and those outside the distinguished subset may, by the inductive hypothesis on w, be expressed in the required form. Thus, by induction, $R(w)$ is true for all w in Z^+, and so the result is established.

(c) By definition, $\pi_1(G) = H_1 = G$. Suppose $i > 1$, and suppose inductively that $\pi_j(G) \subseteq H_j$ for $1 \le j \le i-1$. From (b), $\pi_i(G)$ is generated by elements of the form $\varphi\underline{\pi}^k\underline{\alpha}$ where k is an integer such that $0 \le k \le l$ and φ is a left-normed c-word of c-weight $w(k)$. If $k \ge 1$, then from the definitions, $\text{cpp-wt}\left(\varphi\underline{\pi}^{k-1}\right) = w(k)p^{k-1} < i$. By inductive hypothesis, since $\varphi\underline{\pi}^{k-1}\underline{\alpha} \in \pi_{w(k)p^{k-1}}(G)$, it is also true that $\varphi\underline{\pi}^{k-1}\underline{\alpha} \in H_{w(k)p^{k-1}}$; since the series H is restricted elementary,

$$\varphi\underline{\pi}^k\underline{\alpha} = \varphi\underline{\pi}^{k-1}\underline{\alpha}\underline{\pi} \in H_{w(k)p^k} \subseteq H_i .$$

If $k = 0$, then φ is a left-normed c-word of c-weight i, so $\varphi = [\varphi_1, \varphi_2]$ where $\varphi_1\underline{\alpha} \in \pi_{i-1}(G) \subseteq H_{i-1}$ and $\varphi_2\underline{\alpha} \in G$. Since H is central,

$$\varphi\underline{\alpha} = [\varphi_1\underline{\alpha}, \varphi_2\underline{\alpha}] \in [H_{i-1}, G] \subseteq H_i \ .$$

Since all generators of $\pi_i(G)$ are contained in H_i ,

$$\pi_i(G) \subseteq H_i \ ;$$

and the required result follows by induction.

(d) The proof of part (c) shows that the generators of $\pi_{i+1}(G)$ are of two types: those corresponding to $k = 0$ are of the form $[\varphi_1\underline{\alpha}, \varphi_2\underline{\alpha}] \in [\pi_i(G), G]$, and those corresponding to $k \geq 1$ are of the form

$$(\varphi\underline{\pi}^{k-1}\underline{\alpha})\underline{\pi} \in (\underline{\pi}_{w(k)p^{k-1}}(G))^p .$$

Here $w(k)$ is the least integer such that $w(k)p^k \geq i + 1$, whence

$$w(k)p^k > i \ ,$$

$$w(k)p^{k-1} > [i/p] \ ,$$

and

$$w(k)p^{k-1} \geq [i/p] + 1 = r(i) \ ;$$

so $(\varphi\underline{\pi}^{k+1}\underline{\alpha})\underline{\pi} \in (\underline{\pi}_{r(i)}(G))^p$. This shows that

$$\pi_{i+1}(G) \subseteq \mathrm{sgp}\langle[\pi_i(G), G], (\pi_{r(i)}(G))^p\rangle \ ;$$

the reverse inequality is immediate from the definitions.

(e) Suppose inductively that $\pi_i(G) = \pi_{i+j-1}(G)$ where $i + j - 1 \geq ip$. Note that with $r(i)$ defined as in the statement of (d),

$$i+j-2 \geqslant r(i+j-2) \geq [(ip-1)/p] + 1 = i \ , \qquad \text{and}$$

$$i+j-1 \geqslant r(i+j-1) \geq [ip/p] + 1 = i+1 .$$

Hence $$\pi_{r(i+j-2)}(G) = \pi_{r(i+j-1)}(G) .$$

Now
$$\begin{aligned}\pi_{i+j}(G) &= \operatorname{sgp}\left\langle [\pi_{i+j-1}(G), G], (\pi_{r(i+j-1)}(G))^p \right\rangle \\ &= \operatorname{sgp}\left\langle [\pi_{i+j-2}(G), G], (\pi_{r(i+j-2)}(G))^p \right\rangle \\ &= \pi_{i+j-1}(G) \\ &= \pi_i(G)\end{aligned}$$

The result follows by induction. □

Since the cpp-series is of particular interest in this thesis, some examples are considered. At one extreme, in a group of exponent p, the cpp-series coincides with the lower central series. At another, in the infinite cyclic group Z, for $p^{r-1} < v \le p^r$, the subgroup $\pi_v(Z)$ is the subgroup of index p^r. In yet another direction, in groups such as perfect groups or the quasi-cyclic group Z_{p^∞}, which have no p-elementary abelian homomorphic image, every term of the series is equal to the group itself.

Property (d) is used by Jennings [12], 5.2, to define what he calls the $\underline{\underline{M}}$-series of a finite p-group.

1.7.4 A REFINEMENT OF THE cpp-SERIES

Define, for the $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-word algebra B, and positive integers w and v,

$$\lambda_{w,v}(B) = \{\varphi \in B : \text{either } \text{cpp-wt}(\varphi) > v \text{ or } \text{cpp-wt}(\varphi) = v \text{ and } \text{c-wt } \varphi \ge w\}.$$

This set is easily checked to be an ideal of B, in the sense described in 1.2.4. Let $\underline{\alpha}$ be a surjective homomorphism from B to a group G, and define

$$\lambda_{w,v}(G) = (\lambda_{w,v}(B))\underline{\alpha} .$$

Although $\lambda_{w,v}(B)$ is not a weight ideal, the proof of 1.2.5 still applies, and shows that $\lambda_{w,v}(G)$ is well-defined, independent of the choice of $\underline{\alpha}$. Clearly

$$\pi_v(G) = \lambda_{1,v}(G) \supseteq \lambda_{2,v}(G) \supseteq \ldots \supseteq \lambda_{v,v}(G) \supseteq \lambda_{v+1,v}(G) = \pi_{v+1}(G) .$$

This shows that there are at most v steps introduced between $\pi_v(G)$ and $\pi_{v+1}(G)$; in fact there are considerably fewer.

Suppose that $v = up^s$ where $p \nmid u$. From 1.7.3 (b), it is easily seen that $\pi_v(G)$ is generated by $\pi_{v+1}(G)$ together with the homomorphic images in G of the $s + 1$ words in the set

$$\left\{[\xi_0, \ldots, \xi_{up^t-1}]\underline{\pi}^{s-t} \ : \ 0 \le t \le s\right\} .$$

However,

$$[\xi_0, \ldots, \xi_{up^t-1}]\underline{\pi}^{s-t} \in \gamma_{up^t}(B)\backslash\gamma_{up^t+1}(B) ,$$

whence

$$[\xi_0, \ldots, \xi_{up^t-1}]\underline{\pi}^{s-t} \in \lambda_{up^t,v}(B)\backslash\lambda_{up^t+1,v}(B) .$$

Thus the ideals $\lambda_{i,v}(B)$ such that $1 \le i \le u$ are all equal to $\pi_v(B) = \lambda_{u,v}(B)$, and for $1 \le t \le s$, the ideals $\lambda_{i,v}(B)$ such that $up^{t-1} < i \le up^t$ are all equal to $\lambda_{up^t,v}(B)$. Similar equalities therefore hold for the corresponding verbal subgroups of G. In the refined series, then, there are only $s + 1$ steps between $\pi_v(G)$ and $\pi_{v+1}(G)$; and for $1 \le t \le s$ and $up^{t-1} < w \le up^t$, the subgroup $\lambda_{w,v}(G)$ is generated by $\pi_{v+1}(G)$ and the homomorphic images in G of the $s - t + 1$ words in the set

$$\left\{[\xi_0, \ldots, \xi_{up^r-1}]\underline{\pi}^{s-r} : t \le r \le s\right\} . \qquad \square$$

1.7.5 NILPOTENT AND cpp-NILPOTENT GROUPS

A group G whose lower central series reaches the trivial subgroup after a finite number of steps is called nilpotent. If c is the integer such that

$$\gamma_c(G) \supset \gamma_{c+1}(G) = \{1\}$$

then G is nilpotent of class c .

It is clear that for a finite term of either the lower elementary central series or the cpp-series of G to be the trivial subgroup, it is necessary and sufficient that G be nilpotent and have exponent a power of p . Such a group will be called cpp-nilpotent. The integer d such that

$$\pi_d(G) \supset \pi_{d+1}(G) = \{1\}$$

is called its cpp-class.

The lower central series and the cpp-series are based on the weight functions $w^1_{1,0}$ and $w^p_{1,0}$ respectively. For arbitrary integers a and b such that $a \ge b \ge 0$ and $a \ge 1$, and e in $\{1, p\}$ there exists a similar descending strongly central series based on the weight function $w^e_{a,b}$. Let

$$\gamma_i^{a,b,e}(B) = \left\{\varphi \in B : w^e_{a,b}(\varphi) \ge i\right\}$$

and

$$\gamma_i^{a,b,e}(G) = \gamma_i^{a,b,e}(B)\underline{\alpha}$$

for arbitrary surjective homomorphism $\underline{\alpha} : B \to G$. Note that the subgroup $\varepsilon_i(G)$ defined in 1.7.2 is $\gamma_i^{1,1,1}(G)$.

In a cpp-nilpotent group, a finite term of every such series reaches the trivial subgroup. The largest integer i such that $\gamma_i^{a,b,e}(G) \ne \{1\}$

is called the $\gamma^{a,b,e}$-class of a group G. Note that the $\gamma^{a,b,p}$-class of a group is simply a times its cpp-class; the $\gamma^{a,b,1}$-class is

$$\max\{wa+(s(w)-1)b : 1 \leq w \leq c\}$$

where c is the nilpotency class of the group, and for $1 \leq w \leq c$, the maximum order of a commutator of weight w in G is $p^{s(w)}$.

As was pointed out in the introduction, the invariants just defined play an important role in the calculations of Chapter 3.

CHAPTER 2

A BASIS FOR A GROUP OF PRIME-POWER ORDER

In Chapter 3 of this thesis, groups with normal subgroups of prime-power index will be investigated, and it will be useful to have a standard form in which to express their elements. As a first step, the case in which the normal subgroup is trivial is considered in this chapter.

The crucial result is the construction, in Theorem 2.2.1, of a standard basis for a group of prime-power order, which has, in particular, the property that the standard expression of an arbitrary element relative to this basis is in terms of powers of commutators in a form which exhibits the position of the element in the refined cpp-series of the group. This property is very useful in Chapter 3, and may well have applications in other investigations into the commutator-power structure of prime-power groups.

Throughout this chapter, let p be an arbitrary prime and P a group of p-power order. Let d be the cpp-class of P ; that is,

$$\pi_d(P) \supset \pi_{d+1}(P) = E \ .$$

For each term $\lambda_{w,v}(P)$ of the refined cpp-series of P , let $n(w, v)$ be the integer defined by

$$|\lambda_{w,v}(P)| = p^{n(w,v)} \ ,$$

and, where convenient, let $n(1, v) = n(v)$ so that

$$|\pi_v(P)| = p^{n(v)} \ .$$

Further, let

$$a = 1 + (p-1) \sum \{n(v) : 1 \le v \le d\}$$

and

$$b = (p-1)d \ .$$

Further notation will be described following the definition of a basis.

2.1 Definitions and notation associated with bases

2.1.1 A BASIS

A *basis* for P is a set $\{(\tau_i, h(i)) : i \in M\}$ where M is a finite ordered index set and for i in M, $\tau_i \in P$ and $h(i) \in Z^+$, such that every element of P may be written uniquely in the form

$$\prod\left\{\tau_i^{e(i)} : i \in M\right\}$$

with $0 \le e(i) < p^{h(i)}$ for i in M. This will be called the standard form of the element relative to the basis. □

It was shown by Sylow [27] (Théorème III, p. 588) that every group of order p^n, for n in Z^+, has a basis with $|M| = n$ and $h(i) = 1$ for i in M.

2.1.2 INVARIANTS OF BASES

Let $T = \{(\tau_i, h(i)) : i \in M\}$ be a basis for P. For each i in M, let

$$q(i) = p^{h(i)},$$

$$q^*(i) = p^{h(i)-1}$$

and

$u(i)$ be the integer such that $\tau_i \in \gamma_{u(i)}(P) \backslash \gamma_{u(i)+1}(P)$.

Corresponding to a basis T of P and a term $\lambda_{w,v}(P)$ of the refined cpp-series of P, let

$m(w, v, T)$ be the cardinality of the set of ordered pairs

$\{(i, j) : i \in M, j \in \underline{h}(i)$, and either $u(i)p^j > v$

or $u(i)p^j = v$ and $u(i) \geq w\}$.

(Note that for each pair (i, j) in this set, $\tau_i^{p^j} \in \lambda_{w,v}(P)$.) Denote $m(1, v, T)$ simply as $m(v, T)$.

It is not hard to see that for an arbitrary basis T for P , and a term $\lambda_{w,v}(P)$ of the refined cpp-series,

$$m(w, v, T) \leq n(w, v) .$$

2.1.3 FORM-RESPECTING BASES

A basis T for P is λ-form-respecting if the relation

$$\prod \left\{\tau_i^{e(i)} : i \in M\right\} \in \lambda_{w,v}(P)$$

implies that for each i in M there exists an integer $s(i)$ such that $p^{s(i)} | e(i)$ and either $u(i)p^{s(i)} > v$ or $u(i)p^{s(i)} = v$ with $u(i) \geq w$.

The definition of a π-form-respecting basis is similar; in fact it is equivalent to the weaker condition obtained by restricting w to be 1 in the above.

Clearly a basis T for P is λ-form-respecting if and only if for all terms $\lambda_{w,v}(P)$ of the refined cpp-series of P ,

$$m(w, v, T) = n(w, v) .$$

2.2 Construction of a standard basis

It has not yet been shown that form-respecting bases for all groups of prime-power order exist. The standard basis to be constructed in this section is λ-form-respecting, and has some other useful properties.

2.2.1 THEOREM. *An arbitrary finite p-group P has a basis $T = \{(\tau_i, h(i)) : i \in M\}$ with the properties:*

(i) T is λ-form-respecting;

(ii) M *is ordered in such a way that* $u(i)q(i) \leq u(j)q(j)$ *whenever* i *precedes* j *in* M ; *and*

(iii) *for all* i *in* M, τ_i *can be written as a left-normed commutator of weight* $u(i)$ *in* P .

Proof. The proof proceeds by induction in the reverse direction along the refined cpp-series of P to show that each term $\lambda_{w,v}(P)$ of that series has a basis $T_{w,v} = \left\{\left(\tau_i^{t(w,v,i)},\ h(w,\ v,\ i)\right) : i \in M_{w,v}\right\}$ such that

(a) for all i in $M_{w,v}$, τ_i is a left-normed commutator of weight $u(i)$ in P ;

(b) for all i in $M_{w,v}$, there exists a non-negative integer $r(w,\ v,\ i)$ such that $t(w,\ v,\ i) = p^{r(w,v,i)}$;

(c) for all i in $M_{w,v}$, either

(i) $v \leq u(i)t(w,\ v,\ i) < pv$, or

(ii) $u(i)t(w,\ v,\ i) = pv$ and $u(i) < w$, or

(iii) $t(w,\ v,\ i) = 1$;

(d) if i precedes j in $M_{w,v}$, then

$$u(i)t(w,\ v,\ i)p^{h(w,v,i)} \leq u(j)t(w,\ v,\ j)p^{h(w,v,j)};$$

and

(e) for all $(y,\ x)$ such that $\lambda_{y,x}(P) \subseteq \lambda_{w,v}(P)$,

$$m\left(y,\ x,\ T_{w,v}\right) = n(y,\ x) .$$

Here $m\left(y,\ x,\ T_{w,v}\right)$ is defined to be the cardinality of the set of ordered pairs:

$$\left\{(i,\ j) : i \in M_{w,v},\ j \in \underline{h}(w,\ v,\ i),\ \text{and either } u(i)t(w,\ v,\ i)p^j > x \right.$$
$$\left. \text{or } u(i)t(w,\ v,\ i)p^j = x \text{ and } u(i) \geq y\right\} .$$

Conditions (b) and (c) show that for all i in $M_{1,1}$, $t(1,\ 1,\ i) = 1$;

so that by condition (a), each basis element of $T_{1,1}$ may be written as a left-normed commutator of weight $u(i)$ in P, as required. Condition (d) shows that $M_{1,1}$ is ordered in the way required of M. Finally it has already been noted that condition (e) for the basis $T_{1,1}$ is equivalent to the condition that $T_{1,1}$ is λ-form-respecting. Thus if $T_{1,1}$ is constructed with the properties claimed for it, then the choice $M = M_{1,1}$, $h(i) = h(1, 1, i)$ for each i in M, and hence $T = T_{1,1}$, establishes the truth of the theorem.

The inductive hypothesis is vacuously true for $\lambda_{1,d+1}(P) = E$.

Let $\lambda_{w,v}(P)$ be an arbitrary non-trivial term of the refined cpp-series of P, and suppose the hypothesis established for all terms $\lambda_{w',v'}(P)$ of the series such that either $v' > v$ or $v' = v$ and $w' > w$.

Consider first of all the case $w = v + 1$, and recall that $\lambda_{v+1,v}(P) = \pi_{v+1}(P) = \lambda_{1,v+1}(P)$. By assumption, $\lambda_{1,v+1}(P)$ has a basis $T_{1,v+1}$ with appropriate properties. Let $M_{v+1,v} = M_{1,v+1}$, and for i in $M_{v+1,v}$ let $t(v+1, v, i) = t(1, v+1, i)$ and $h(v+1, v, i) = h(1, v+1, i)$. Then $T_{v+1,v}$ is identical with $T_{1,v+1}$, and so is a basis for $\lambda_{v+1,v}(P)$. Clearly $T_{v+1,v}$ inherits properties (a), (b), (d) and (e) from $T_{1,v+1}$; that it also inherits property (c) will now be shown. All indices i in $M_{1,v+1}$ satisfying condition (c) (iii) for $T_{1,v+1}$ satisfy the corresponding condition for $T_{v+1,v}$. No index i can satisfy condition (c) (ii) for $T_{1,v+1}$, since $u(i) \geq 1$ for all i in $M_{1,v+1}$. So if $t(1, v+1, i) \neq 1$, then $v + 1 \leq u(i)t(1, v+1, i) < p(v+1)$. Also by condition (b), $p | t(1, v+1, i)$, whence

$$v + 1 \leq u(i)t(1, v+1, i) \leq p(v+1) - p = pv .$$

If $u(i)t(1, v+1, i) < pv$, then i satisfies condition (c) (i) for $T_{v+1,v}(P)$. Otherwise,

$$u(i)t(1, v+1, i) = pv$$

and

$$u(i) = pv/t(1, v+1, i) \leq pv/p < v + 1 ,$$

so that condition (c) (ii) for $T_{v+1,v}(P)$ is satisfied.

A step of the type considered involves only a change in notation. Consider now the case $1 \leq w \leq v$, where a basis $T_{w+1,v}$ for $\lambda_{w+1,v}(P)$ is known to exist and to have appropriate properties. As the second subscript on each symbol T, M, t, r, h and m remains equal to v throughout the following construction, it will be omitted for the sake of reducing the notational complexity. The construction of T_w now takes place in two stages.

In the first stage, let

$$\Gamma = \{i \in M_{w+1} : u(i) = w, u(i)t(w, i) = pv\}$$

and note that for i in Γ, $t(w, i) \geq p$ since $w \leq v$. For i in Γ , let $r(w, i) = r(w+1, i) - 1$ and $h(w, i) = h(w+1, i) + 1$, and for i in $M_{w+1}\backslash\Gamma$, let $r(w, i) = r(w+1, i)$ and $h(w, i) = h(w+1, i)$. For all i in M_{w+1} , let $t(w, i) = p^{r(w,i)}$. Now define

$$T^*_w = \left\{\left(\tau_i^{t(w,i)}, h(w, i)\right) : i \in M_{w+1}\right\} .$$

Let $\lambda^*_{w,v}(P)$ be the set of elements of P which may be written in the form $\prod\left\{\tau_i^{t(w,i)e(i)} : i \in M_{w+1}\right\}$, where for i in M_{w+1} , $0 \leq e(i) < p^{h(w,i)}$. Then $\lambda^*_{w,v}(P)$ contains $\lambda_{w+1,v}(P)$ and is a subgroup of P ; in fact a normal subgroup, since $\lambda^*_{w,v}(P)$ is contained in $\lambda_{w,v}(P)$, and

$$\lambda_{w,v}(P)/\lambda_{w+1,v}(P) \text{ is central in } P/\lambda_{w+1,v}(P)\ .$$

If $\lambda^*_{w,v}(P)$ were equal to $\lambda_{w,v}(P)$, then M_w would be chosen equal to M_{w+1}, and the construction of $T_w = T^*_w$ would be complete. Otherwise, subsection 1.7.4 shows that the elementary abelian subgroup $\lambda_{w,v}(P)/\lambda^*_{w,v}(P)$ is generated by cosets each having a representative which is a homomorphic image in P of the word $[\xi_0, \ldots, \xi_{w-1}]\underline{\pi}^r$ in the $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-word algebra B, where $wp^r = v$. In the second stage of the construction, a subset $\left\{\tau_i^{t(i)} : i \in M_w \backslash M_{w+1}\right\}$ of these representatives is selected in such a way that the corresponding cosets constitute a basis for $\lambda_{w,v}(P)/\lambda^*_{w,v}(P)$. Here M_w is an ordered set such that all elements of $M_w \backslash M_{w+1}$ precede all elements of M_{w+1}. For i in $M_w \backslash M_{w+1}$, let $u(i) = w$ (the weight of the left-normed commutator τ_i in P), let $t(w, i) = p^{r(w,i)} = p^r$, and let $h(w, i) = 1$. Then the construction of

$$T_w = \left\{\left(\tau_i^{t(w,i)},\ h(w,\ i)\right)\ :\ i \in M_w\right\}$$

is complete.

The next stage of the proof is to show that T^*_w is a basis for $\lambda^*_{w,v}(P)$. For this, it is sufficient to show that the elements $\left\{\tau_i^{t(w,i)} : i \in \Gamma\right\}$ are independent modulo $\lambda_{w+1,v}(P)$, that is, to show that if

$$(*) \qquad \prod\left\{\tau_i^{t(w,i)e(i)}\ :\ i \in \Gamma\right\} \in \lambda_{w+1,v}(P)$$

with $0 \le e(i) < p$ for each i in Γ, then $e(i) = 0$ for each i in Γ. Law (v) for group-like varieties shows that

$$\left[\prod\left\{\tau_i^{t(w,i)e(i)}\ :\ i \in \Gamma\right\}\right]^p = \prod\left\{\tau_i^{pt(w,i)e(i)}\ :\ i \in \Gamma\right\}\prod\left\{\psi_d\ :\ d \in \Delta\right\}$$

d/ where for $\cancel{e}$ in Δ, $\psi_d \in \lambda_{pw,pv}(P) \subseteq \lambda_{w+1,pv}(P)$. From the same law,

$$\left(\lambda_{w+1,v}(P)\right)^p \subseteq \lambda_{w+1,pv}(P) \ ;$$

hence from the relation (*) it follows that

$$\prod\left\{\tau_i^{pt(w,i)e(i)} : i \in \Gamma\right\} = \prod\left\{\tau_i^{t(w+1,i)e(i)} : i \in \Gamma\right\} \in \lambda_{w+1,pv}(P) \ .$$

Now property (e) of T_{w+1} shows that for all i in Γ , $p|e(i)$; so if $0 \le e(i) < p$, then $e(i) = 0$, as required.

The fact that T_w is a basis for $\lambda_{w,v}(P)$ is now an immediate consequence of the fact that

$$\left\{\prod\left\{\tau_i^{t(w,i)e(i)} : i \in M_w \backslash M_{w+1}\right\} : 0 \le e(i) < p \ \text{ for } i \in M_w \backslash M_{w+1}\right\}$$

is a complete set of coset representatives for $\lambda^*_{w,v}(P)$ in $\lambda_{w,v}(P)$.

The final stage of the proof is a verification that the basis T_w has the properties (a) to (e) listed at the start of this proof. It is clear from the construction that (a), (b) and (d) are satisfied. For j in Γ or in $M_w \backslash M_{w+1}$, it is clear that $u(j)t(w, j) = v$, so that condition (c) (i) is satisfied. Those j in M_{w+1} which satisfy condition (c) (i) or (c) (iii) for T_{w+1} clearly satisfy the same condition for T_w , and those which satisfy condition (c) (ii) for T_{w+1} either satisfy the same condition for T_w or are contained in Γ .

It is clear from the way in which the basis elements were constructed that

$$m\left(w, v, T_{w,v}\right) = n(w, v) \ ;$$

and if $\lambda_{y,x}(P) \subseteq \lambda_{w+1,v}(P)$, then

$$\begin{aligned} m\left(y, x, T_{w,v}\right) &= m\left(y, x, T_{w+1,v}\right) \\ &= n(y, x) \ , \end{aligned}$$

so that condition (e) is also satisfied. □

CHAPTER 3

GROUP EXTENSIONS

Throughout this chapter, G is an arbitrary group with a normal subgroup H whose index in G is a power of an arbitrary but fixed prime p. Further notation and definitions are introduced in Section 3.1. Among the concepts defined there are standard commutators and standard commutator segments; their properties are investigated in Section 3.2. Section 3.3 then leads up to the main result in Theorem 3.3.3 about upper bounds on the nilpotency and cpp-classes of extensions of cpp-nilpotent groups by finite p-groups. In Section 3.4, these bounds are shown to be best possible by the fact that they equal corresponding lower bounds on the classes of a wreath product satisfying the same conditions.

3.1 Definitions and notation

3.1.1 A STANDARD BASIS FOR G MODULO H

Throughout this chapter, let a and b be the values taken for the finite p-group G/H by the parameters defined in the introduction to Chapter 2, and let c and d be respectively the nilpotency class and the cpp-class of G/H.

A *basis for* G *modulo* H is defined to be an ordered set $\{(\tau_i, h(i)) : i \in M\}$ such that every element of G may be expressed uniquely in the form $\prod\{\tau_i^{e(i)} : i \in M\}\eta$, where $\eta \in H$ and for i in M, $0 \le e(i) < p^{h(i)}$. In this situation, the results of Chapter 2 apply, with the obvious minor modifications. Throughout this chapter, let $T = \{(\tau_i, h(i)) : i \in M\}$ be a λ-form-respecting basis for G modulo H with the property that for i in M, τ_i is a left-normed commutator of weight $u(i)$ in G, ordered in such a way that if i precedes j in M, then $u(i)p^{h(i)} \le u(j)p^{h(j)}$.

For i in M, set

$$q(i) = p^{h(i)} \quad \text{and} \quad q^*(i) = p^{h(i)-1} .$$

Let z be the last element of M.

Let S be the set of "permissible powers" of basis elements; that is,

$$S = \left\{\tau_i^{p^r} : i \in M \text{ and } r \in \underline{h}(i)\right\} .$$

For $1 \le v \le d$, let S_v be the subset of S consisting of elements whose cpp-weight in G is at least v; that is,

$$S_v = \left\{\tau_i^{p^r} : i \in M,\ r \in \underline{h}(i), \text{ and } u(i)p^r \ge v\right\} .$$

3.1.2 STANDARD COMMUTATORS

In the situation already described, a *standard commutator* is defined to be a left-normed commutator

$$[\rho, \vartheta_1, \ldots, \vartheta_n]$$

such that $\rho \in H$ and for $1 \le l \le n$, $\vartheta_l \in S$.

A *standard commutator segment* is defined to be a finite sequence of elements of S. It may be written in the form

$$\chi = \tau_{i(1)}^{p^{r(1)}}, \tau_{i(2)}^{p^{r(2)}}, \ldots, \tau_{i(m)}^{p^{r(m)}}$$

where for $1 \le l \le m$, $i(l) \in M$ and $r(l) \in \underline{h}\big(i(l)\big)$. If ρ is an element of H and χ_1 and χ_2 are standard commutator segments, then

$$[\rho, \chi_1] \quad \text{and} \quad [\rho, \chi_1, \chi_2]$$

are the standard commutators obtained simply by writing the appropriate sequences of entries in place of χ_1 and χ_2.

The *weight* of the standard commutator segment

$$\chi = \tau_{i(1)}^{p^{r(1)}}, \tau_{i(2)}^{p^{r(2)}}, \ldots, \tau_{i(m)}^{p^{r(m)}}$$

is defined to be

$$w(\chi) = \sum \left\{u\left(i(l)\right)p^{r(l)} : 1 \le l \le m\right\} .$$

The *profile* of the same standard commutator segment χ is defined to be a function f from the set $\{v \in \mathsf{Z} : 1 \le v \le d\}$ to N such that for $1 \le v \le d$, the segment χ has precisely $f(v)$ entries of cpp-weight v in G; that is, there are $f(v)$ integers l such that $1 \le l \le m$ and $u\left(i(l)\right)p^{r(l)} = v$. Note that

$$\sum \{f(v) : 1 \le v \le d\} = m$$

and

$$\sum \{vf(v) : 1 \le v \le d\} = w(\chi) .$$

A profile f_1 is said to be *heavier* than a profile f_2 if for some integer v such that $1 \le v \le d$,

$$v < v' \le d \Rightarrow f_1(v') = f_2(v')$$

and

$$f_1(v) > f_2(v) .$$

The relation "is heavier than or equal to" is easily seen to be a total, reflexive, transitive, and antisymmetric relation on the set of all profiles, and thus to be a linear order. It is in fact a reversal of a well-ordering; the inductive argument on profile in the proof of Lemma 3.2.7 involves only a finite process.

3.1.3 SOME WEIGHT IDEALS OF H

The integers a and b defined in 3.1.1 satisfy the relation $a > b \ge 0$. Hence the definitions in 1.7.5 of subgroups $\gamma_l^{a,b,e}(H)$ are applicable.

In particular, if l and m are positive integers, define

$$\gamma(l,\ m) = \gamma_l^{a,b,1}(H) \cap \gamma_m^{a,b,p}(H)\ .$$

Note that:

$$\gamma(a,\ a) = H\ ,$$

$$[\gamma(l,\ m),\ \gamma(l',\ m')] \subseteq \gamma(l+l',\ m+m')\ ,$$

and

$$\gamma(l,\ m)^p \subseteq \gamma(l+b,\ m+a)\ .$$

3.2 Manipulation of standard commutators

Throughout this section, the notation and terminology already described are used. To avoid repetition, it is also to be assumed in each lemma and corollary that l and m are positive integers, each at least a, and that $\rho \in \gamma(l,\ m)$.

The first lemma in this section adapts law (iv) for group-like varieties to the situation considered here, showing that "commutation" with a standard commutator segment distributes over multiplication modulo an appropriate subgroup.

3.2.1 LEMMA. *Let l and m be positive integers, and $\{\rho_g : g \in \Gamma\}$ and $\{\chi_g : g \in \Gamma\}$ ordered sets, respectively of elements of $\gamma(l,\ m)$ and of standard commutator segments. Let χ also be a standard commutator segment. Then*

$$\left[\prod \{[\rho_g,\ \chi_g] : g \in \Gamma\},\ \chi\right] = \prod \{[\rho_g,\ \chi_g,\ \chi] : g \in \Gamma\}\zeta$$

where $\zeta \in \gamma(l+a,\ m+a)$.

Proof. Note that for g in Γ, $\rho_g \in H = \gamma(a,\ a)$; hence each commutator in G with two entries from the set $\{\rho_g : g \in \Gamma\}$ is contained in $L = \gamma(l+a,\ m+a)$. Suppose $\chi = \vartheta_1, \ldots, \vartheta_k$; and proceed by induction on k . When $k = 0$, the result holds trivially. When $k > 0$, let $\chi^* = \vartheta_1, \ldots, \vartheta_{k-1}$; and assume inductively that the result for χ^* is established. Then

$$\begin{aligned}
&\left[\prod\{[\rho_g, \chi_g] : g \in \Gamma\}, \chi\right] \\
&\quad = \left[\prod\{[\rho_g, \chi_g] : g \in \Gamma\}, \chi^*, \vartheta_k\right] \\
&\quad \equiv \left[\prod\{[\rho_g, \chi_g, \chi^*] : g \in \Gamma\}, \vartheta_k\right] \pmod{L} \quad \text{(by inductive hypothesis)} \\
&\quad \equiv \prod\{[\rho_g, \chi_g, \chi^*, \vartheta_k] : g \in \Gamma\} \pmod{L} \quad \text{(by law (iv))} \\
&\quad = \prod\{[\rho_g, \chi_g, \chi] : g \in \Gamma\} ,
\end{aligned}$$

as required. □

When the last entry in an otherwise standard commutator is a product, then this commutator is equal, modulo an appropriate subgroup, to a suitable product of standard commutators. Though not quite a "distributive law from the left", this result is related to the preceding one.

3.2.2 LEMMA. *Let* $\vartheta \in \pi_v(G)$. *Then*

$$[\rho, \vartheta] = \prod\{[\rho, \chi_g] : g \in \Gamma\}\zeta$$

where for g *in* Γ , χ_g *is a non-empty standard commutator segment each of whose entries* $\tau_i^{p^j}$ *satisfies* $u(i)p^j \geq v$, *and where* $\zeta \in \gamma(l+a,\ m+a)$.

Proof. From the definition of the basis T ,

$$\vartheta = \prod\left\{\tau_k^{e(k)} : k \in M\right\}\eta$$

where $\eta \in H$, and for k in M there exists a non-negative integer $s(k)$ such that $p^{s(k)} | e(k)$ and $u(k)p^{s(k)} \geq v$. The result follows by repeated use of the fact that (from Lemma 1.6.1 (a))

$$[\rho, \zeta_1\zeta_2 \ldots \zeta_n] = [\rho, \zeta_2 \ldots \zeta_n][\rho, \zeta_1][\rho, \zeta_1, \zeta_2 \ldots \zeta_n] ,$$

and by noting that $[\rho, \eta] \in \gamma(l+a, m+a)$. □

The next lemma and its corollary investigate rearrangement of the order of entries in a standard commutator segment.

3.2.3 LEMMA. *Let* ϑ *and* τ *be elements of* G . *Then*

$$[\rho, \tau, \vartheta] = [\rho, \vartheta, \tau][\rho, [\tau, \vartheta]][\rho, \tau, [\tau, \vartheta]] [\rho, \vartheta, [\tau, \vartheta]][\rho, \vartheta, \tau, [\tau, \vartheta]]\zeta$$

where $\zeta \in \gamma(l+a, m+a)$.

Proof. As before, note that each commutator in G with two entries equal to ρ is in $L = \gamma(l+a, m+a)$; and work modulo this subgroup. Then

$$\begin{aligned}
[\rho, \tau, \vartheta] &= \rho^{-\tau}\rho\rho^{-\vartheta}\rho^{\tau\vartheta} \\
&\equiv \rho^{-\vartheta}\rho\rho^{-\tau}\rho^{\vartheta\tau}\rho^{-\vartheta\tau}\rho^{\vartheta\tau[\tau,\vartheta]} \pmod L \\
&= [\rho, \vartheta, \tau][\rho^{\vartheta\tau}, [\tau, \vartheta]] \\
&= [\rho, \vartheta, \tau][\rho[\rho, \tau][\rho, \vartheta][\rho, \vartheta, \tau], [\tau, \vartheta]] \\
&\equiv [\rho, \vartheta, \tau][\rho, [\tau, \vartheta]][\rho, \tau, [\tau, \vartheta]][\rho, \vartheta, [\tau, \vartheta]] \\
&\qquad [\rho, \vartheta, \tau, [\tau, \vartheta]] \pmod L ,
\end{aligned}$$

as required. □

3.2.4 COROLLARY. *Let* χ *be a standard commutator segment* $\chi = \vartheta_1, \ldots, \vartheta_k$, *and* $\underline{\delta}$ *be a permutation of the set* $\{j \in Z^+ : 1 \le j \le k\}$. *Then there exists a set* $\{\chi_g : g \in \Gamma\}$ *of standard commutator segments such that*

$$[\rho, \chi] = [\rho, \vartheta_{1\underline{\delta}}, \ldots, \vartheta_{k\underline{\delta}}] \prod \{[\rho, \chi_g] : g \in \Gamma\}\zeta$$

where for g *in* Γ , $w(\chi_g) \ge w(\chi)$ *and the profile of* χ_g *is strictly heavier than that of* χ ; *and where* $\zeta \in \gamma(l+a, m+a)$.

Proof. The permutation $\underline{\delta}$ is a product of transpositions of adjacent integers. For such a transposition, say $(j, j+1)$, let χ_1' be the segment preceding the jth entry, ϑ_j' say, and χ_2' the segment following the $j+1$th entry, ϑ_{j+1}', of a standard commutator segment χ' which is already obtained from χ by a permutation. Apply Lemma 3.2.3 to the standard commutator $[\rho, \chi_1', \vartheta_j', \vartheta_{j+1}']$, and note that each factor outside $\gamma(l+a, m+a)$ in the resulting expression, except $[\rho, \chi_1', \vartheta_{j+1}', \vartheta_j']$, has strictly heavier profile. Then apply Lemma 3.2.1 to show that $[\rho, \chi]$ is equal, modulo $\gamma(l+a, m+a)$, to a product of $[\rho, \chi_1', \vartheta_{j+1}', \vartheta_j', \chi_2']$ with other standard commutators, each having a first entry equal to ρ, and the remaining segment of weight at least $w(\chi)$ and profile strictly heavier than that of χ.

A repetition of this argument for each transposition factor of $\underline{\delta}$ gives the required result. □

The next lemma is related to law (vi) for group-like varieties. It is the first step in showing that under appropriate circumstances, a given standard commutator may be expressed as a product of others in which the first entry is "heavier" and the remaining segment may be correspondingly "lighter".

3.2.5 LEMMA. *For arbitrary* ϑ *in* G,

$$[\rho, p\vartheta] = [\rho, \vartheta^p][\sigma, \vartheta]\zeta$$

where $\sigma \in \gamma(l+b, m+a)$ *and* $\zeta \in \gamma(l+a, m+a)$.

Proof. Note that every commutator in G with two or more entries equal to ρ is contained in $L = \gamma(l+a, m+a)$. It is easy to prove by induction on n, using 1.6.1 (a), that for all n in Z^+,

$$[\rho, \vartheta^n] \equiv \prod \{[\rho, j\vartheta]^{\binom{n}{j}} : 1 \le j \le n\} \pmod{L},$$

where $\binom{n}{j}$ is the binomial coefficient. Hence, and from law (iv) for a group-like variety,

$$[\rho, \vartheta^p] \equiv \left[\prod\{[\rho, (j-1)\vartheta]^{\binom{p}{j}} : 1 \le j \le p-1\}^{-1}, \vartheta\right]^{-1}[\rho, p\vartheta] \pmod{L} ;$$

and since for $1 \le j \le p-1$, $p \mid \binom{p}{j}$, it follows that

$$\sigma = \prod\{[\rho, (j-1)\vartheta]^{\binom{p}{j}} : 1 \le j \le p-1\}^{-1} \in \gamma(l+b, m+a) ,$$

as required.

3.2.6 COROLLARY. *Suppose that a standard commutator segment* χ *has at least* p *entries equal to an element* $\tau_i^{p^r}$ *of* S. *Then there exist a set* $\{\chi_g : g \in \Gamma\} \cup \{\omega\}$ *of standard commutator segments, and elements* σ *in* $\gamma(l+b, m+a)$ *and* ζ *in* $\gamma(l+a, m+a)$ *such that*

$$[\rho, \chi] = \prod\{[\rho, \chi_g] : g \in \Gamma\}[\sigma, \omega]\zeta$$

where

$$w(\chi) - b \le w(\omega) < w(\chi) ,$$

and for each g *in* Γ, $w(\chi_g) \ge w(\chi)$ *and the profile of* χ_g *is strictly heavier than the profile of* χ, *differing from it on some integer strictly greater than* $u(i)p^r$.

Proof. By Corollary 3.2.4, if $L = \gamma(l+a, m+a)$, then there exists a set $\{\chi_g : g \in \Gamma_1\}$ of standard commutator segments, each with weight at least as great as that of χ and with heavier profile, such that

$$[\rho, \chi] \equiv [\rho, \chi'] \prod\{[\rho, \chi_g] : g \in \Gamma_1\} \pmod{L} .$$

Here χ' is obtained from χ by a permutation bringing p entries each equal to $\tau_i^{p^r}$ to the first p places; and from the proof of Lemma 3.2.3 it can be seen that for each g in Γ_1, the profile of χ_g differs from that of χ on some integer strictly greater than $u(i)p^r$. Suppose that

$$\chi' = \underbrace{\tau_i^{p^r}, \ldots, \tau_i^{p^r}}_{p \text{ entries}}, \chi'' .$$

Lemmas 3.2.5 and 3.2.2 now show that

$$[\rho, \chi'] \equiv \left[\rho, \tau_i^{p^{r+1}}, \chi''\right]\left[\sigma, \tau_i^{p^r}, \chi''\right] \pmod{L}$$

where $\sigma \in \gamma(l+b, m+a)$ and

$$w\left(\tau_i^{p^r}, \chi''\right) = w(\chi') - (p-1)u(i)p^r ,$$

whence, setting $\omega = \tau_i^{p^r}, \chi''$,

$$w(\chi) - b \leq w(\omega) < w(\chi)$$

as required. If $r + 1 \neq h(i)$, then the first factor is also of the required form, since

$$w\left(\tau_i^{p^{r+1}}, \chi''\right) = w(\chi)$$

and the profile of $\tau_i^{p^{r+1}}, \chi''$ is strictly heavier than that of χ, differing from it on the integer $u(i)p^{r+1}$. However if $r + 1 = h(i)$, then $\tau_i^{p^{r+1}} \notin S$. In this case, note that $\tau_i^{p^{r+1}} \in \pi_{u(i)p^{r+1}}(G)$, and apply Lemma 3.2.2 to show that

$$\left[\rho, \tau_i^{p^{r+1}}\right] \equiv \prod \{[\rho, \chi_g] : g \in \Gamma_2\} \pmod{L}$$

where for g in Γ_2 , χ_g is a non-empty standard commutator segment each of whose entries has weight at least $u(i)p^{r+1}$. (Of course, if $\tau_i^{p^{r+1}} \in H$ then Γ_2 is empty, and by convention the empty product is the identity.)

By Lemma 3.2.1,

$$\left[\rho,\ \tau_i^{p^{r+1}},\ \chi''\right] \equiv \prod \{[\rho,\ \chi_g,\ \chi''] : g \in \Gamma_2\} \pmod{L} ,$$

and each factor in the last product is easily checked to be of the required form. □

3.2.7 LEMMA. *Let* l *and* m *be positive integers,* $\rho \in \gamma(l, m)$ *, and* χ *be a standard commutator segment such that* $w(\chi) = a$ *. Then there exist sets* $\{\rho_g : g \in \Gamma\}$ *of elements of* H *and* $\{\chi_g : g \in \Gamma\}$ *of standard commutator segments, and an element* ζ *of* $\gamma(l+a, m+a)$ *such that*

$$[\rho, \chi] = \prod \{[\rho_g, \chi_g] : g \in \Gamma\}\zeta$$

and for g *in* Γ *,*

$$\rho_g \in \gamma(l+b, m+a)$$

and

$$w(\chi) - b \le w(\chi_g) \le a - 1 .$$

Proof. Proceed by induction on the profile of χ . If χ has more than $(p-1)n(d)$ entries whose weight is d , then there exists a pair (i, r) where $i \in M$, $r \in \underline{h}(i)$, and $u(i)p^r = d$, such that χ has at least p entries equal to $\tau_i^{p^r}$. Note that in the proof of Lemma 3.2.3, if either ϑ or τ is contained in $\pi_d(G)$ then $[\vartheta, \tau] \in H$ and $[\sigma^{\vartheta\tau}, [\vartheta, \tau]] \in L = \gamma(l+a, m+a)$. Hence, as in the proof of Corollary 3.2.4,

$$[\rho, \chi] \equiv \left[\rho,\ p\tau_i^{p^r},\ \chi'\right] \pmod{L}$$

where the segment $p\tau_i^{p^r}$, χ' is obtained from χ by a permutation of its entries. Since $\tau_i^{p^{r+1}} \in H$ and the weight of $\tau_i^{p^r}$, χ' is $w(\chi) - (p-1)d = w(\chi) - b$, Lemma 3.2.5 now gives the required result.

If each entry of weight d in χ occurs no more than $p - 1$ times, then χ has profile f such that $f(d) \le (p-1)n(d)$. Suppose inductively that the Lemma is proved for all standard commutator segments whose profiles are heavier than that of χ. From the hypotheses of the lemma, there exists a pair (i, r) with $i \in M$ and $r \in \underline{h}(i)$ such that there are at least p entries in χ equal to $\tau_i^{p^r}$. Among such pairs, choose a particular one (i, r) such that the integer $u(i)p^r$ is maximal.

All required results now follow by Corollary 3.2.6 and the inductive hypothesis. □

3.3 An upper bound on the nilpotency class and cpp-class of some group extensions

Theorem 3.3.3 is the main result of this thesis, already referred to in the introduction. The preceding subsections assemble other results needed for its proof.

3.3.1 LEMMA (Frobenius; Kaloujnine and Krasner). *Let M be an arbitrary group with a normal subgroup N. Then $N \operatorname{Wr} M/N$ has a subgroup isomorphic with M.* □

Here $N \operatorname{Wr} M/N$ is the unrestricted standard wreath product of the two groups, as defined by Huppert [12], Kapitel I, 15.6 (p. 97) and by Hanna Neumann [22] at the beginning of section 2.2 (p. 45). The result stated above is Theorem 22.21 (pp. 46-47) in the latter reference; a simple proof and further references are given there.

3.3.2 LEMMA. *Let G be a group which splits over a normal subgroup H of index a power of p. If c is the nilpotency class of G/H, then every left-normed commutator in G of weight l strictly greater than c*

may be expressed as a product of standard commutators of the form

$$\prod\{[\rho_g, \chi_g] : g \in \Gamma\}$$

where there exists m *in* Z^+ *such that for all* g *in* Γ ,

$$\rho_g \in \gamma(m, m) ,$$

$$m + w(\chi_g) \geq l + a - 1 ,$$

and

$$w(\chi_g) \leq a - 1 .$$

Proof. Let J be a complement of H in G . Corresponding to each element α of H , there exists a unique pair (ϑ, η) in $J \times H$ such that $\alpha = \vartheta\eta$. Proceed by induction on l to prove the proposition that for all l in Z^+ , if for $1 \leq i \leq l$, $\alpha_i = \vartheta_i\eta_i$, then

$$[\alpha_1, \ldots, \alpha_l] = [\vartheta_1, \ldots, \vartheta_l] \prod\{[\rho_g, \chi_g] : g \in \Gamma_l\}$$

where Γ_l is an ordered index set, and for g in Γ_l there exists an integer $m(g)$ such that $\rho_g \in \gamma(m(g), m(g))$ and $m(g) + w(\chi_g) \geq l + a - 1$, and $w(\chi_g) \leq a - 1$.

When $l = 1$, the result is true since $\eta_1 \in \gamma(a, a)$. Suppose $l > 1$, and the result for $l - 1$ established. Then, by law (iv) for group-like varieties,

$$\begin{aligned}
&[\alpha_1, \ldots, \alpha_{l-1}, \alpha_l] \\
&\quad = \Big[[\vartheta_1, \ldots, \vartheta_{l-1}] \prod\{[\rho_g, \chi_g] : g \in \Gamma_{l-1}\}, \vartheta_l\eta_l\Big] \\
&\quad = [\vartheta_1, \ldots, \vartheta_{l-1}, \vartheta_l][\vartheta_1, \ldots, \vartheta_{l-1}, \eta_l] \prod\{[\rho_g, \chi_g, \vartheta_l] : g \in \Gamma_{l-1}\} \\
&\qquad\qquad \prod\{[\rho_g, \chi_g, \eta_l] : g \in \Gamma_{l-1}\} \prod\{\zeta_d : d \in \Delta_l\} .
\end{aligned}$$

In this product the first factor $[\vartheta_1, \ldots, \vartheta_{l-1}, \vartheta_l]$ is that required by the hypothesis. The second factor is trivial if $l - 1 > c$, and otherwise is the inverse of $[\eta_l, [\vartheta_1, \ldots, \vartheta_{l-1}]]$, which by Lemma 3.2.2 is a product of standard commutators of the required form. For g in Γ_{l-1}, the commutator $[\rho_g, \chi_g, \vartheta_l]$ is also of the required form by Lemma 3.2.2 unless it happens that some factors in the resulting product have standard commutator segments whose weight exceeds $a - 1$; however each of these may in turn be expressed in the required form by Lemma 3.2.7. Since for g in Γ_{l-1}, $\rho_g \in \gamma(l-1, l-1)$, each term of the form $[\rho_g, \chi_g, \eta_l]$ is contained in $\gamma(l+a-1, l+a-1)$, and hence is of the required form $[\rho_d, \chi_d]$ with trivial segment χ_d. The same argument applies to each ζ_d for d in Δ_l, since each such element is a commutator with at least two entries from the set $\{\rho_g : g \in \Gamma_{l-1}\}$.

The proposition stated at the beginning of the proof now follows by induction, and so the Lemma itself is proved. □

3.3.3 THEOREM. *If G is a group with a normal subgroup H such that for some prime p, the index of H is a power of p and H is cpp-nilpotent, then G is nilpotent (and hence cpp-nilpotent). If a and b are the values for G/H of the invariants defined in the introduction to Chapter 2, then the nilpotency class of G is bounded above by the $w^1_{a,b}$-class of H, and the cpp-class of G is bounded above by the $w^p_{a,b}$-class of H.*

Proof. Let K be the base group of the wreath product $W = H \operatorname{Wr} G/H$. Then K is a direct power of H, and for e in $\{1, p\}$, has the same $w^e_{a,b}$-class as H; and $W/K \cong G/H$. Further, since W splits over K, Lemma 3.3.2 shows that every left-normed commutator of weight l in W, where l is greater than the nilpotency class of G/H, may be expressed as a product of standard commutators $[\rho, \chi]$ where ρ is contained in

$$\gamma_l^{a,b,1}(K) \cap \gamma_l^{a,b,p}(K) .$$

Thus every left-normed commutator in W whose weight exceeds the $w^1_{a,b}$-class of K and hence of H is trivial, and by 1.7.1 (b) the nilpotency class of W is bounded above by the $w^1_{a,b}$-class of H. Similarly, the p^rth power of a left-normed commutator of weight l in W is contained in $\gamma^{a,b,p}_{lp^r}(K)$; whence it follows by 1.7.3 (b) that the cpp-class of W is bounded above by the $w^p_{a,b}$-class of K and hence of H.

Since G is isomorphic with a subgroup of W, by Lemma 3.3.1, it follows that the same upper bounds apply to the nilpotency class and the cpp-class of G. □

3.4 The classes of a wreath product

In Lemma 3.4.4, lower bounds on the nilpotency class and cpp-class of a wreath product are obtained. Since they are equal to the corresponding upper bounds given by Theorem 3.3.3, they show that the result is best possible, and give (Corollary 3.4.5) the exact nilpotency class and cpp-class of a wreath product.

This result is briefly compared with earlier upper and lower bounds on the class of a nilpotent wreath product.

3.4.1 COEFFICIENTS

Let h be a positive integer, $q = p^h$ (where of course p is the same arbitrary prime already being considered), and $q^* = p^{h-1}$. Let C be the cyclic group of order q generated by τ, and ZC the integer group ring of C.

Define $R(q, x, k)$ to be the coefficient of τ^k in the standard expression for $(-1+\tau)^x$ in $\mathbb{Z}C$; that is,

$$(-1+\tau)^x = \sum \{R(q, x, k)\tau^k : k \in \underline{q}\} .$$

This coefficient is, apart from sign, the same as the integer $\lambda_{x,k}$ defined by Liebeck in [16], 4.1 and 4.2; note that the choice of sign is not consistently maintained between 4.1 and 4.2. Liebeck's Theorem 4.3 still is valid, and is quoted now in the notation of the present thesis:

3.4.2 LEMMA (Liebeck, [16], Theorem 4.3). *Let* s *be a positive integer.*

(a) If $x \geq q + (s-1)(q-q^*)$ *, then*

$$p^s \mid R(q, x, k) \text{ for all } k \text{ in } \underline{q} .$$

(b) If $x = q + s(q-q^*) - 1$ *, then*

$$p^{s+1} \nmid R(q, x, k) \text{ for all } k \text{ in } \underline{q} . \quad \square$$

3.4.3 LEMMA. *Let* $W = H \text{ Wr } J$ *be the wreath product of two groups, and let* $\rho \in H(1)$ *, the first coordinate subgroup in the base group of* W *, and* $\tau \in J$ *where* τ *has order* $q = p^h$ *. Then for arbitrary* l *in* $\mathbb{Z}^+$ *,*

$$[\rho, l\tau] = \prod \{\rho^{R(q,l,k)\tau^k} : k \in \underline{q}\} .$$

Proof. Elements of the form ρ^{τ^i} and ρ^{τ^j} commute, since if $i \not\equiv j \pmod q$ then they belong to distinct coordinate subgroups in the base group of W . An element of the form $\prod \{\rho^{n(k)\tau^k} : k \in \underline{q}\}$ may be regarded as ρ^χ where $\chi = \sum \{n(k)\tau^k : k \in \underline{q}\}$ is an element of the integer group ring $\mathbb{Z}C$ of the cyclic group C of order q generated by τ . Note that for all χ in $\mathbb{Z}C$,

$$[\rho^\chi, \tau] = \rho^{\chi(-1+\tau)} .$$

Hence

$$[\rho,\ l\tau] = \rho^{(-1+\tau)^{l}} = \prod\{\rho^{R(q,l,k)\tau^{k}} : k \in \underline{q}\}\ ,$$

as required. $\square$

3.4.4 THEOREM. *Let J be a group whose order is a power of a prime p, on which the invariants defined in the introduction to Chapter 2 take the values a and b. Let H be a cpp-nilpotent group, with $w^{e}_{a,b}$-class $l(e)$ for e in $\{1, p\}$. Then in the wreath product $H \operatorname{Wr} J$, there exist*

(a) a non-trivial commutator of weight $l(1)$, and

(b) a non-trivial scpp-element of cpp-weight $l(p)$.

Proof. The construction used here is essentially the same as that used by Teresa Scruton [26], Theorem 3.5, though in the present context more detailed argument is required.

(a) From the hypothesis, and property 1.7.1 (b), there exists in H a non-trivial element $\kappa^{p^{s}}$ where $\kappa = [\alpha_1, \ldots, \alpha_w]$ is a left-normed commutator of weight w in H, and where

$$aw + bs = l(1)\ .$$

Use the same symbols κ and α_i, $1 \le i \le w$, to denote the corresponding elements in the first coordinate subgroup $H(1)$ of the base group K of $W = H \operatorname{Wr} J$.

Let $T = \{(\tau_i, h(i)) : i \in M\}$ be a standard basis for J as constructed in Section 2.2, and for i in M let $q(i) = p^{h(i)}$. Denote by ω the standard commutator segment

$$\underbrace{\tau_1, \ldots, \tau_1}_{q(1)-1},\ \underbrace{\tau_2, \ldots, \tau_2}_{q(2)-1},\ \ldots,\ \underbrace{\tau_z, \ldots, \tau_z}_{q(z)-1}$$

and by ω^* the standard commutator segment

$$\underbrace{\tau_1, \ldots, \tau_1}_{q(1)-1}, \ldots, \underbrace{\tau_{z-1}, \ldots, \tau_{z-1}}_{q(z-1)-1}$$

where z is the last element of the ordered set M.

From some j in M, suppose that $\nu = \prod\{\nu(\iota) : \iota \in J\}$ is the standard expression, in terms of the direct product, for an element of K such that $\nu(\iota) \neq 1$ only if ι is of the form

$$\prod\left\{\tau_i^{f(i)} : i \in M \text{ and } i < j\right\}.$$

Then $\nu' = \left[\nu, \left(p^{h(j)}-1\right)\tau_j\right]$ has a standard expression in which $\nu'(\iota) \neq 1$ only if ι is of the form $\prod\left\{\tau_i^{f(i)} : i \in M \text{ and } i \leq j\right\}$; and the component of ν' in the first coordinate subgroup is $\nu'(1) = \nu(1)^e$ where $e = (-1)^{p^{h(j)}-1}$.

Hence the standard commutator $[\alpha_1, \omega]$ whose weight in G is precisely a, has either α_1 or α_1^{-1} as its component in the first coordinate subgroup of K; and the commutator $[\alpha_1, \omega, \alpha_2]$ has respectively either $[\alpha_1, \alpha_2]$ or $\left[\alpha_1^{-1}, \alpha_2\right]$ as its component in the first coordinate subgroup, and 1 as its component in each other coordinate subgroup. Similarly, by a repetition of this procedure, the commutator

$$[\alpha_1, \omega, \alpha_2, \omega, \ldots, \alpha_w, \omega^*]$$

whose weight in G is $aw - u(z)\left(q(z)-1\right)$, has

$$\left[\cdots\left[\alpha_1^e, \alpha_2\right]^e, \ldots, \alpha_w\right]^{e'}$$

(where e and e' are in the set $\{1, -1\}$) as its component in the first coordinate subgroup, and 1 as its component in each coordinate subgroup $H(\iota)$ indexed by an element ι in J such that $\iota = \prod\left\{\tau_i^{f(i)} : i \in M\right\}$ with $f(z) \neq 0$, where z is the last element of M. By Lemma 3.4.3, if

$$r = s\left(p^{h(z)}-p^{h(z)-1}\right) = s\left(q(z)-q^*(z)\right),$$

then the commutator

$$\nu = \left[\alpha_1, \omega, \ldots, \alpha_w, \omega, r\tau_z\right]$$

$$= \left[\alpha_1, \omega, \ldots, \alpha_w, \omega^*, \left(r+q(z)-1\right)\tau_z\right]$$

whose weight in G is $aw + b(s-1) = l(1)$, has

$$\left[\ldots\left[\alpha_1^e, \alpha_2\right]^e, \ldots, \alpha_w\right]^{e'R\left(q(z),x,0\right)}$$

as its component in the first coordinate subgroup, where

$$x = q(z) \quad + \quad s\left(q(z)-q^*(z)\right) \quad - \quad 1 .$$

By Lemma 3.4.2,

$$p^s \mid R\left(q(z), x, 0\right) \quad \text{and} \quad p^{s+1} \nmid R\left(q(z), x, 0\right) .$$

By hypothesis, $\left[\alpha_1, \alpha_2, \ldots, \alpha_w\right]^{p^s} \neq 1$, but every element of H whose $w^1_{a,b}$-weight is greater is trivial. Hence

$$\left[\ldots\left[\alpha_1^e, \alpha_2\right]^e, \ldots, \alpha_w\right]^{e'R(q,x,0)} = \left[\alpha_1, \alpha_2, \ldots, \alpha_w\right]^{e''R(q,x,0)}$$

(where $e'' \in \{1, -1\}$), which is non-trivial. That is, the commutator ν has non-trivial component in the first coordinate subgroup of K , so is itself non-trivial, as required.

(b) Similarly, from the hypothesis and property 1.7.4 (b), there exists in H a non-trivial element of the form κ^{p^s} where $\kappa = \left[\alpha_1, \ldots, \alpha_w\right]$ is a left-normed commutator of weight w in H and $awp^s = l(p)$. As in the proof of part (a), the commutator

$$\nu = \left[\alpha_1, \omega, \alpha_2, \omega, \ldots, \alpha_w, \omega\right]$$

of weight aw in G has, as its component in the first coordinate subgroup of K ,

$$\left[\ldots\left[\alpha_1^e, \alpha_2\right]^e, \ldots, \alpha_w\right]^e .$$

Hence the scpp-element ν^{p^s} whose cpp-weight in G is $awp^s = l(p)$, has

$$\left[\ldots\left[\alpha_1^e, \alpha_2\right]^e, \ldots, \alpha_w\right]^{ep^s} = [\alpha_1, \alpha_2, \ldots, \alpha_w]^{e'p^s},$$

which is non-trivial, as its component in the first coordinate subgroup of K, and so is itself non-trivial, as required. □

3.4.5 COROLLARY. *If J is a group whose order is a power of a prime p, with invariants a and b as described in Chapter 2, and if H is a cpp-nilpotent group for the same prime p, such that the $w_{a,b}^1$-class of H is $l(1)$ and the $w_{a,b}^p$-class of H is $l(p)$, then the nilpotency class and cpp-class of $W = H \operatorname{Wr} J$ are precisely $l(1)$ and $l(p)$ respectively.* □

3.4.6 COMPARISONS AND COMMENTS

If H has nilpotency class r, and for $1 \le w \le r$, the maximum order of a commutator of weight w in r is $p^{s(w)}$, then the $w_{a,b}^1$-class of H is

$$\max\{aw+b(s(w)-1) : 1 \le w \le r\}.$$

Hence the above expression is the nilpotency class of $H \operatorname{Wr} J$ where J is a finite p-group with invariants a and b.

The lower bound given by Teresa Scruton in Theorem 3.5 of [26] may be expressed in similar form,

$$\max\{a_S w+b_S(s(w)-1) : 1 \le w \le r\},$$

where $p^{s(w)}$ is the exponent of $\gamma_w(H)$, $a_S = 1 + t(p-1)$ where p^t is the order of J, and $b_S = p - 1$. When J is elementary abelian, a_S and b_S are equal to a and b respectively; so in this case her result is exact for all H, though she only claims this for H abelian. In fact, at first sight there appears to be a conflict, since for some w the exponent of $\gamma_w(H)$ may be greater than the order of any individual commutator of weight w. However, the $p^{s(w)-1}$th power of an element of $\gamma_w(H)$ has $w^1_{a,b}$-weight at least $aw + b(s(w)-1)$, and so by Corollary 1.6.3 may be expressed as a product of scpp-elements of H whose $w^1_{a,b}$-weight is at least as great; thus though for some w, $1 \le w \le r$, the expressions may differ, their maximum values must be equal.

J.D.P. Meldrum [18], considering only the case where J is abelian, obtains a similar expression

$$\max\{aw+b(s_M(w)-1) \,:\, 1 \le w \le r\}$$

for the class of $H \operatorname{Wr} J$. His definition of $s_M(w)$ is a little complicated, but with the help of Corollary 1.6.3 it is easily seen that the maximum value of the above expression occurs for an integer w such that $p^{s_M(w)}$, the wth exponent of H, is precisely the maximum order of a commutator of weight w in H.

Larry Morley on the other hand considers in [20] the case where J is not necessarily abelian, and H is. He works in terms of a basis for J related to a central series, but not so suitable for the purpose as that in Chapter 2 of this thesis. The integer a given here is essentially one more than $(p-1)$ times the sum of the cpp-weights of elements in a refined π-respecting basis. For the cpp-weight, the corresponding sum in Morley's upper bound substitutes the product of the exponents of higher factors in the central series. His thesis [19] gives a lower bound, not contained in [20], in which, roughly speaking, the integer 1 is substituted for the cpp-weight. This is improved, for class two groups, in [21]. In the thesis, both upper and lower bounds are extended to the case where H is not abelian.

Robert Sandling points out in [23], (Lemma 1.2 and Proposition 1.11) that the last part of Theorem 3.7 in Jennings paper [13] gives the exact nilpotency class (namely, a in the notation of the present chapter) of $C_p \text{ Wr } J$ for all finite p-groups J. Before the statement of Theorem 1.3 of [23] he advances the conjecture that for arbitrary positive integer e and (non-trivial) finite p-group J, the nilpotency class of $C_{p^{e-1}} \text{ Wr } J$ is strictly less than that of $C_{p^e} \text{ Wr } J$. If $e \geq 2$, then the former is $a + b(e-2)$ and the latter $a + b(e-1)$; and since whenever J is non-trivial, $b \geq p-1 \geq 1$, the truth of the conjecture follows. The "equivalent statements" of Theorem 1.3 in [23] thus all become theorems.

CHAPTER 4

LOWER BOUNDS ON SOME POTENTIAL "RESTRICTED BURNSIDE" GROUPS

The restricted Burnside problem is the question whether, among all finite groups with d generators and exponent n, there is a largest. Such a group, if it exists, is denoted $\overline{B}(d, n)$. The unrestricted Burnside problem is whether the free group $B(d, n)$ on d generators with exponent n is finite. Clearly if it is finite, then it provides a positive answer to the restricted problem.

It is known that $B(d, n)$ is finite for all d when n is 2 (trivial), 3 (Burnside [5]; the correct orders are given by Levi and van der Waerden [15]), 4 (Sanov [25]; the orders are largely unknown) or 6 (M. Hall [8]; the orders are given by the work of Hall and Higman [10] on the restricted problem for exponent 6); and Kostrikin has shown [14] that $\overline{B}(d, n)$ exists for all d whenever n is prime.

The application to electronic computers of the nilpotent quotient algorithm described by Macdonald [17] and Wamsley [28], and adapted by M.F. Newman (unpublished) to the lower p-elementary central series of a group, gives a tool which has the potential of proving, for some prime powers q and integers d, that $\overline{B}(q, d)$ exists, and of giving a presentation which exhibits its order and class. This method has been used by Bayes, Kautsky and Wamsley [3] to give a presentation for $B(3, 4)$ showing that it has order 2^{69} and class 7, by Wamsley in a paper by Havas, Wall and Wamsley [11] to give a presentation for $\overline{B}(2, 5)$ (order 5^{34} and class 12), and by Alford, Havas and Newman [1] to give a presentation for $B(4, 4)$ (order 2^{422} and class 10).

An interesting question raised is whether this method will settle such unsolved questions as whether there exist groups $\overline{B}(2, 8)$, $\overline{B}(2, 9)$ and

$\overline{B}(2, 25)$. Already prospects appear gloomy, given the present capacities of computers and the presently-forseen capacities of programmes, since large lower bounds on the orders of these groups are well-known, though unpublished. Let F_n be the free group of rank n . For arbitrary group G and variety $\underline{\underline{V}}$, let $\underline{\underline{V}}(G)$ be the intersection of all normal subgroups N of G such that $G/N \in \underline{\underline{V}}$; that is, $\underline{\underline{V}}(G)$ is the minimal normal subgroup of G such that $G/\underline{\underline{V}}(G) \in \underline{\underline{V}}$. Where $G = F_n$, denote $G/\underline{\underline{V}}(G)$ as $F_n(\underline{\underline{V}})$. The Schreier formula for the rank of a normal subgroup of a free group shows that

$$\underline{\underline{B}}_3(F_2) \text{ has rank } 3^3 + 1 = 28 ,$$

$$\underline{\underline{B}}_4(F_2) \text{ has rank } 2^{12} + 1 = 4097 ,$$

and

$$\underline{\underline{B}}_5(F_2) \text{ has rank } 5^{34} + 1 \simeq 6 \times 10^{23} ;$$

whence

$$F_2(\underline{\underline{B}}_3\underline{\underline{B}}_3) \text{ has order } 3^{3+28+\binom{28}{2}+\binom{28}{3}} = 3^{3685} ,$$

$$F_2(\underline{\underline{A}}_2\underline{\underline{B}}_4) \text{ has order } 2^{12+4097} = 2^{4109} ,$$

and

$$F_2(\underline{\underline{A}}_5\underline{\underline{B}}_5) \text{ has order } 5^{34+5^{34}+1} \approx 5^{6\times 10^{23}} .$$

The product varieties $\underline{\underline{B}}_3\underline{\underline{B}}_3$, $\underline{\underline{A}}_2\underline{\underline{B}}_4$, and $\underline{\underline{A}}_5\underline{\underline{B}}_5$ are of course proper subvarieties of $\underline{\underline{B}}_9$, $\underline{\underline{B}}_8$ and $\underline{\underline{B}}_{25}$ respectively.

Prospects of computing these groups are made even more gloomy by the high nilpotency classes of some two-generator groups in the varieties $\underline{\underline{B}}_3\underline{\underline{B}}_3$, $\underline{\underline{A}}_2\underline{\underline{B}}_4$, and $\underline{\underline{B}}_5\underline{\underline{B}}_5$ (18, 39 , and 11,244 respectively). It should be emphasised that there is no reason for believing that these lower bounds are "good"; indeed there is some for suspecting the contrary. The group

$F_2(\underline{\underline{B}}_3\underline{\underline{B}}_3)$ may well have class nearer the upper bound of 27 than the lower bound of 18 ; and although the class of $F_2(\underline{\underline{A}}_2\underline{\underline{B}}_4)$ is easily shown to be precisely 39 , the class of $F_2(\underline{\underline{B}}_4\underline{\underline{B}}_4 \cap \underline{\underline{B}}_8)$ appears to be much greater, and may well be over a hundred (rough calculations suggest 123 as an upper bound).

4.1 Some examples of nilpotent wreath products

4.1.1 EXAMPLES

(a) $B(2, 3)$ Wr $B(2, 3)$ has nilpotency class 18

The group $B(2, 3)$, generated by α and β , has a standard basis:

$$\tau_1 = \alpha \qquad h_1 = 1$$

$$\tau_2 = \beta \qquad h_2 = 1$$

$$\tau_3 = [\beta, \alpha] \qquad h_3 = 1 ;$$

and it is easy to see that $m(1) = 3$, $m(2) = 1$,

$$a = 1 + (p-1)(3+1) = 9$$

and

$$b = (p-1)2 = 4 .$$

The same group considered as the bottom group has $w^1_{9,4}$-class 18 .

(b) $B(3, 3)$ Wr $B(3, 3)$ has nilpotency class 75

A standard basis for $B(3, 3)$ with generators α, β and γ is:

$$\tau_1 = \alpha \qquad \tau_2 = \beta \qquad \tau_3 = \gamma$$

$$\tau_4 = [\beta, \alpha] \qquad \tau_5 = [\gamma, \alpha] \qquad \tau_6 = [\gamma, \beta]$$

$$\tau_7 = [\beta, \alpha, \gamma]$$

with $h(i) = 1$ for $1 \le i \le 7$. Hence

$$m(1) = 7 , \quad m(2) = 4 , \quad m(3) = 1 ,$$

$$a = 1 + (p-1)(7+4+1) = 25 ,$$

and

$$b = (p-1)3 = 6 .$$

group

Considered as the bottom in the wreath product, $B(3, 3)$ has $w^1_{25,6}$-class $25 \times 3 = 75$.

(c) $C_2 \times C_2$ Wr $B(2, 4)$ has nilpotency class 39

Information about $B(2, 4)$ is easily read off from the presentation for $B(3, 4)$ given by Bayes, Kautsky, and Wamsley [3]. If $B(2, 4)$ has generators α and β , then a λ-form-respecting basis is given by:

$$\begin{array}{ll}
\tau_1 = \alpha & h(1) = 2 \\
\tau_2 = \beta & h(2) = 2 \\
\tau_3 = [\beta, \alpha] & h(3) = 1 \\
\tau_4 = [\beta, \alpha, \alpha, \alpha] & h(4) = 1 \\
\tau_5 = [\beta, \alpha, \alpha, \beta] & h(5) = 1 \\
\tau_6 = [\beta, \alpha, \beta, \beta] & h(6) = 1 \\
\tau_7 = [\beta, \alpha, \alpha] & h(7) = 2 \\
\tau_8 = [\beta, \alpha, \beta] & h(8) = 2 .
\end{array}$$

From this it can be seen that

$$m(1) = 12 , \quad m(2) = 10 , \quad m(3) = 7 ,$$
$$m(4) = 5 , \quad m(5) = 2 , \quad m(6) = 2 ;$$

whence

$$a = 1 + (12+10+7+5+2+2) = 39$$

and

$$b = 6 .$$

Clearly the $w^1_{39,6}$-class of $C_2 \times C_2$ is 39 .

(d) $B(2, 4)$ Wr $B(2, 4)$ has nilpotency class 195 and cpp-class 234

Though this fact is not apparent from the λ-form-respecting basis shown above, the nilpotency class of $B(2, 4)$ is 5 , and

$$w^1_{39,6}\big(B(2, 4)\big) = 39 \times 5 = 195 .$$

It is however, apparent above that

$$w^2_{39,6}\big(B(2, 4)\big) = 39 \times 6 = 234 .$$

(e) $\overline{B}(2, 5)$ Wr $\overline{B}(2, 5)$ has nilpotency class 11244

A presentation for $\overline{B}(2, 5)$ is given in the paper by Havas, Wall, and Wamsley [11]. From this, it can be calculated that:

$$\begin{array}{lll} m(1) = 34 , & m(2) = 32 , & m(3) = 31 , \\ m(4) = 29 , & m(5) = 26 , & m(6) = 24, \\ m(7) = 20 , & m(8) = 16 , & m(9) = 12 , \\ m(10) = 6 , & m(11) = 3 , & m(12) = 1 . \end{array}$$

Hence

$$a = 1 + 4 \times 234 = 937 ,$$

and

$$b = 4 \times 12 = 48 .$$

The $w^1_{937,48}$-class of $\overline{B}(2, 5)$ is $12 \times 937 = 11244$. □

4.2 Adaptation of earlier results

In a standard wreath product, the results of Section 3.2 may be specialised. If ρ is taken to be an element of a coordinate subgroup, then the element ζ in $\gamma(l+a, m+a)$ in the statements of the lemmas of Section 3.2 becomes the identity; alternatively, equalities replace congruences modulo this subgroup.

Let $W = H \text{ Wr } J$, let K be the base group of W , and let T be a standard basis for a copy of J in W . The notation of Chapter 3, apart from the assumption that ρ is in $\gamma(l, m)$, will be used.

4.2.1 LEMMA. *If* ρ *is an element of a coordinate subgroup of* W, *and* χ_1 *and* χ_2 *are arbitrary standard commutator segments, then*

$$[[\rho, \chi_1], [\rho, \chi_2]] = 1 .$$

Proof. For arbitrary standard commutator segment χ, the component of $[\rho, \chi]$ in each coordinate subgroup of K is a power of ρ. All such components, whether or not in distinct coordinate subgroups, commute with one another. □

4.2.2 LEMMA (*cf.* Lemma 3.2.1). *Let* ρ *be an element of a coordinate subgroup of* W, *and let* χ *and* χ_g *for* g *in* Γ *be standard commutator segments. Then*

$$\left[\prod \{[\rho, \chi_g] : g \in \Gamma\}, \chi\right] = \prod \{[\rho, \chi_g, \chi] : g \in \Gamma\} .$$

Proof. Proceed by induction on the number of entries of χ, as in the proof of 3.2.1. The congruences modulo L in that proof become equalities, because of Lemma 4.2.1. □

4.2.3 LEMMA (*cf.* Lemma 3.2.2). *Let* $\rho \in K$ *and* $\vartheta \in \pi_v(J)$. *Then*

$$[\rho, \vartheta] = \prod \{[\rho, \chi_g] : g \in \Gamma\}$$

where for g *in* Γ, χ_g *is a non-empty standard commutator segment each of whose entries is in* S_v.

Proof. As in Lemma 3.2.2, noting that in the wreath product, $\eta = 1$. □

4.2.4 LEMMA. *Let* ρ *be an element of a coordinate subgroup of* W, *let* χ *be a standard commutator segment* $\chi = \vartheta_1, \ldots, \vartheta_k$, *and let* $\underline{\delta}$ *be a permutation of the set* $\{j \in Z^+ : 1 \le j \le k\}$. *Then there exists a set* $\{\chi_g : g \in \Gamma\}$ *of standard commutator segments, each with weight at least as* *great as that of* χ *and with profile strictly heavier than that of* χ *, such that*

$$[\rho, \chi] = [\rho, \vartheta_{1\underline{\delta}}, \ldots, \vartheta_{k\underline{\delta}}] \prod \{[\rho, \chi_g] : g \in \Gamma\} .$$

Proof. Use Lemma 4.2.1 in the proofs of 3.2.3 and 3.2.4. □

4.2.5 LEMMA. *If* ρ *is an element of a coordinate subgroup of* W *, and if* $\vartheta \in S$ *, then*

$$[\rho, p\vartheta] = [\rho, \vartheta^p][\sigma^p, \vartheta]$$

where σ *is a product of conjugates of* ρ *by elements of* J *.*

Proof. Use 4.2.1 in the proof of 3.2.5. □

4.2.6 LEMMA. *Let* ρ *be an element of a coordinate subgroup of* W *, let* v *be an integer such that* $1 \le v \le d$ *, and let* χ *be a standard commutator segment with at least* $p - 1$ *entries equal to each element of* S_{v+1} *and* p *entries equal to* $\tau_i^{p^r}$ *where* $u(i)p^r = v$ *. Then*

$$[\rho, \chi] = \prod \left\{\left[\rho_g^p, \chi_g\right] : g \in \Gamma\right\}$$

where for g *in* Γ *,* ρ_g *is a product of conjugates of* ρ *by elements of* J *, and* $w(\chi_g) \ge w(\chi) - b$ *.*

Proof. Proceed by induction in the reverse direction on v . If $v = d$, then for arbitrary ϑ in J , $\left[\vartheta, \tau_i^{p^r}\right] = 1$; hence, using 4.1.1 in the proof of 3.2.3,

$$[\rho, \chi] = [\rho, \chi']$$

where

$$\chi' = \underbrace{\tau_i^{p^r}, \ldots, \tau_i^{p^r}}_{p \text{ entries}}, \chi'' ,$$

and χ' is obtained from χ by a permutation. Since $\left(\tau_i^{p^r}\right)^p = 1$, Lemma 4.2.5 shows that

$$\left[\rho,\ p\tau_i^{p^r}\right] = \left[\sigma^p,\ \tau_i^{p^r}\right],$$

and Lemma 4.2.2 then gives the required result.

If $v < d$, then suppose inductively that the result is proved for all integers greater than v. Now Lemma 4.2.4 gives

$$[\rho, \chi] = [\rho, \chi'] \prod \{[\rho, \chi_g] : g \in \Gamma_1\}$$

where χ' has the same form as in the initial case, and for g in Γ_1, χ_g has strictly heavier profile than has χ; from consideration of the proof of Lemma 3.2.3, it can be seen that χ_g has strictly more entries from S_{v+1} than has χ. Thus each factor $[\rho, \chi_g]$ for g in Γ_1 may, by the inductive hypothesis, be expressed in the required form. Lemmas 4.2.5, 4.2.3 and 4.2.2, together with the inductive hypothesis, now give the required result. □

4.3 Two-generator subgroups

4.3.1 LEMMA. *Let* p *be a prime,* m *a positive integer, and* J *a finite two-generator group of exponent* p^m *with a* λ*-form-respecting basis* $T = \{(\tau_i, h(i)) : i \in M\}$ *such that, where* ρ *and* σ *generate* J,

$$\tau_1 = \rho, \quad \tau_2 = \sigma,$$

and for each i *in* M, τ_i *is a left-normed commutator with* $u(i)$ *entries from the set* $\{\rho, \sigma\}$. *Let* H *be a* nilpotent *two-generator group of exponent* p. *Then the standard wreath product* $W = H \text{ Wr } J$ *has a two-generator subgroup whose nilpotency class is the same as that of* W.

Proof. Let H be generated by α and β, and have nilpotency class r. Let a and b be the parameters for J defined in Chapter 2. The nilpotency class and cpp-class of W are both equal to ar; and from the proof of Theorem 3.4.4, with the standard commutator segment ω defined as it was there, it can be seen that there exists a non-trivial commutator

$$\nu = \left[\beta,\ (p^{h(1)}-1)\tau_1,\ (p^{h(2)}-1)\tau_2,\ \ldots,\ (p^{h(z)}-1)\tau_z,\ \alpha,\ \ldots\right]$$

$$= [\beta,\ \omega,\ \alpha,\ \omega,\ \ldots,\ \omega]$$

of weight ar in W. The goal of this proof is the construction of a non-trivial commutator of the same weight ar in a two-generator subgroup of W.

Let X be the subgroup of W generated by the elements $\rho\alpha$ and $\sigma\beta$. Let μ be the element of X obtained by substituting for each entry of ν equal to either α or ρ the entry $\rho\alpha$ and for each entry of ν equal to either β or σ the entry $\sigma\beta$. For example, if $\tau_3 = [\sigma, \rho]$, then

3/ $$\mu = [\sigma\beta,\ (p^{h(1)}-1)\rho\alpha,\ (p^{h(2)}-1)\sigma\beta,\ (p^{h(3)}-1)[\sigma\beta,\ \rho\alpha],\ \ldots,\ \rho\alpha,\ \ldots] .$$

Since ν has maximal c-weight and cpp-weight in W, application of law (iv) shows that

$$\mu = \prod \{\mu_g : g \in \Gamma\}$$

where $|\Gamma| = 2^{ar}$ and for each g in Γ, μ_g is a commutator with the same bracketting arrangement as ν and μ, and precisely ar entries: either α or ρ in each position where ν has either α or ρ, and either β or σ in each position where ν has either β or σ. One such commutator μ_g is equal to ν, and so is non-trivial; it will be shown that all the others are trivial.

If a commutator μ_g for g in Γ has among the entries in its first segment of weight a (corresponding to the first $[\beta, \omega]$ in ν) two entries from the set $\{\alpha, \beta\}$, then this segment is contained in $\gamma(2a, 2a)$ in the notation defined in 3.1.3. Hence, by a similar procedure to that in the proof of Lemma 3.3.2,

$$\mu_g \in \gamma(ar+1, ar+1) = \{1\} .$$

A similar argument shows that if, for some integer w such that $1 \le w \le r$, the initial segment by μ_g with weight aw contains $w + 1$ entries from $\{\alpha, \beta\}$, then $\mu_g = 1$.

On the other hand, if μ_g has an initial segment of weight greater than c with no entry from the set $\{\alpha, \beta\}$, it is clearly again equal to the identity.

Hence the initial segment of weight a in each non-trivial commutator μ_g for g in Γ must contain one and only one entry from the set $\{\alpha, \beta\}$. If this entry, η say, is not in the first position, then it may be brought there as follows. If η occurs replacing an entry other than the first in a commutator

$$\tau_j = [\xi_1, \ldots, \xi_{i-1}, \xi_i, \xi_{i+1}, \ldots] ,$$

then

$$\begin{aligned} \eta^* &= [\xi_1, \ldots, \xi_{i-1}, \eta, \xi_{i+1}, \ldots] \\ &= \Big[[\eta, [\xi_1, \ldots, \xi_{i-1}]]^{-1}, \xi_{i+1}, \ldots\Big] . \end{aligned}$$

In any case, a similar manipulation gives

$$\begin{aligned} \mu_g &= [\sigma, \vartheta_1, \ldots, \vartheta_{l-1}, \eta^*, \vartheta_{l+1}, \ldots] \\ &= [\eta^*, [\sigma, \vartheta_1, \ldots, \vartheta_{l-1}], \vartheta_{l+1}, \ldots]^{-1} . \end{aligned}$$

If η was originally the second entry, then this process gives a commutator

$$[\alpha, \sigma, (q(1)-2)\rho, (q(2)-1)\sigma, \ldots]^{-1}$$

with an initial segment of weight a having $q(2) = p^{h(2)}$ entries equal to σ and, for each i in M with $u(i) \geq 2$, having $q(i) - 1$ $(= p^{h(i)}-1)$ entries equal to τ_i . Since H has exponent p , repeated applications of Lemma 4.2.5 show that μ_g is equal to a commutator with p entries equal to $\tau_2 = \sigma$ and $p - 1$ entries equal to each element of the set S_2 ;

hence by Lemma 4.2.6 and the fact that H has exponent p, it follows that $\mu_g = 1$.

Similarly, if η was originally in a position later than the second, and if $[\sigma, \vartheta_1, \ldots, \vartheta_{l-1}]$ has weight v in G, then Lemma 4.2.3 together with a repetition of the above argument referring to S_{v+1} rather than to S_2 shows again that $\mu_g = 1$.

Hence each non-trivial factor μ_g for g in Γ must have β as its first entry, and this must be followed immediately by the standard commutator segment ω of weight $a - 1$.

standard The commutator μ_g cannot have a commutator segment of weight a, however, for every such commutator may be brought, by Lemma 4.2.5, to a form shown by Lemma 4.2.6 to be trivial. Hence if μ_g is non-trivial, the first entry in each of its r segments of weight a must be from $\{\alpha, \beta\}$ and each other entry must be from $\{\rho, \sigma\}$; that is, $\mu_g = \nu$. □

4.3.2 COROLLARY. *There exist in the product varieties* $\underline{\underline{B}}_3\underline{\underline{B}}_3$, $\underline{\underline{A}}_2\underline{\underline{B}}_4$ *and* $\underline{\underline{B}}_5\underline{\underline{B}}_5$ *two-generator groups of nilpotency classes* 18, 39 *and* 11244 *respectively.*

Proof. This is immediate from the examples in 4.1.1 and Lemma 4.3.1. □

The significance of this Corollary has been discussed in the introduction to Chapter 4.

REFERENCES

[1] William A. Alford, George Havas and M.F. Newman, "Groups of exponent 4 ", *Notices Amer. Math. Soc.* 22 (1975), A-301.

[2] Gilbert Baumslag, "Wreath products and p-groups", *Proc. Cambridge Philos. Soc.* 55 (1959), 224-231.

[3] A.J. Bayes, J. Kautsky and J.W. Wamsley, "Computation in nilpotent groups (application)", *Proc. Second Internat. Conf. Theory of Groups*, Canberra, 1973 (Lecture Notes in Mathematics, 372, 82-89. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

[4] Claude Berge, *Graphs and Hypergraphs* (translated by Edward Minieka. North-Holland, Amsterdam, London; American Elsevier, New York, 1973).

[5] W. Burnside, "On an unsettled question in the theory of discontinuous groups", *Quart. J. Math.* 33 (1902), 230-238.

[6] P.M. Cohn, *Universal Algebra* (Harper and Row, New York, 1965).

[7] William Haebich, "The multiplicator of various products of groups", PhD thesis, Australian National University, Canberra, 1972.

[8] Marshall Hall, Jr., "Solution of the Burnside problem for exponent 6 ", *Illinois J. Math.* 2 (1958), 764-786.

[9] P. Hall, "A contribution to the theory of groups of prime-power order", *Proc. London Math. Soc.* (2) 36 (1933), 29-95.

[10] P. Hall and Graham Higman, "On the p-length of p-soluble groups and reduction theorems for Burnside's problem", *Proc. London Math. Soc.* (3) 6 (1956), 1-42.

[11] George Havas, G.E. Wall, and J.W. Wamsley, "The two generator restricted Burnside group of exponent five", *Bull. Austral. Math. Soc.* 10 (1974), 459-470.

[12] B. Huppert, *Endliche Gruppen I* (Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[13] S.A. Jennings, "The structure of the group ring of a p-group over a modular field", *Trans. Amer. Math. Soc.* 50 (1941), 175-185.

[14] А.И. Кострикин [A.I. Kostrikin], "О проблеме Бернсайда" [The Burnside problem], *Izv. Akad. Nauk SSSR Ser. Mat.* 23 (1959), 3-34; *Amer. Math. Soc. Transl.* (2) 36 (1964), 63-99.

[15] Friedrich Levi and B.L. van der Waerden, "Über eine besondere Klasse von Gruppen" [On a particular class of groups], *Abh. Math. Sem. Univ. Hamburg* 9 (1933), 154-158.

[16] Hans Liebeck, "Concerning nilpotent wreath products", *Proc. Cambridge Philos. Soc.* 58 (1962), 443-451.

[17] I.D. Macdonald, "A computer application to finite p-groups", *J. Austral. Math. Soc.* 17 (1974), 102-112.

[18] J.D.P. Meldrum, "On nilpotent wreath products", *Proc. Cambridge Philos. Soc.* 68 (1970), 1-15; "Correction to a paper on wreath products", *Proc. Cambridge Philos. Soc.* 76 (1974), 21.

[19] Larry John Morley, "Bounds for the class of nilpotent wreath products", PhD thesis, University of Illinois, 1969.

[20] Larry Morley, "Bounds on the nilpotency class of certain semidirect products", *Trans. Amer. Math. Soc.* 159 (1971), 381-390.

[21] Larry J. Morley, "A commutator relation in a wreath product of p-groups. Preliminary report", *Notices Amer. Math. Soc.* 22 (1975), A-675.

[22] Hanna Neumann, *Varieties of Groups* (Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[23] Robert Sandling, "Modular augmentation ideals", *Proc. Cambridge Philos. Soc.* 71 (1972), 25-32.

[24] Robert Sandling, "Subgroups dual to dimension subgroups", *Proc. Cambridge Philos. Soc.* 71 (1972), 33-38.

[25] И.Н. Санов [I.N. Sanov], "Решение проблемы Бернсайда для показателя 4 " [Solution of Burnside's Problem for exponent 4], *Leningrad Gos. Univ. Ped. Inst. Uč. Zap. Mat. Ser.* 10 (1940), 166-170.

[26] Teresa Scruton, "Bounds for the class of nilpotent wreath products", *Proc. Cambridge Philos. Soc.* 62 (1966), 165-169.

[27] M.L. Sylow, "Théorèmes sur les groupes de substitutions", *Math. Ann.* 5 (1872), 584-594.

[28] J.W. Wamsley, "Computation in nilpotent groups (theory)", *Proc. Second Internat. Conf. Theory of Groups*, Canberra, 1973 (Lecture Notes in Mathematics, 372, 691-700. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

[29] M.A. Ward, "Basic commutators", *Philos. Trans. Roy. Soc. London Ser. A* **264** (1969), 343-412.